AN INTRODUCTION

TO

SPHERICAL AND PRACTICAL

ASTRONOMY.

BY

DASCOM GREENE,

Professor of Mathematics and Astronomy in the Rensselaer Polytechnic Institute.

GINN & COMPANY

BOSTON · NEW YORK · CHICAGO · LONDON

38.4

The Athenæum Press
GINN & COMPANY · PROPRIETORS · BOSTON · U.S.A.

PREFACE.

The present volume is the result of an attempt to supply the want of a text-book adapted to the needs of students who wish to begin the study of spherical and practical astronomy, and who are prepared to do so by a sufficient acquaintance with the several branches of mathematics, and with the general principles of astronomy. It claims to be no more than an introduction to the subject, and aims to present its first principles in an elementary and practical form for the use of beginners. Accordingly, many topics which would properly find a place in a more extended treatise are here but partially developed or perhaps omitted entirely. For the same reason, no elaborate computations are attempted, the examples given being of such simple kind as best serve to illustrate the use of the working formulæ.

The plan of the work embraces only those practical methods which can be carried out by the use of portable instruments. The fact that it is primarily intended for use in the class-room, and not for field use, will explain the omission of many details connected with the making of observations, which, while important, can only be learned by the actual handling of instruments.

The Appendix contains an elementary exposition of the

method of least squares, with practical applications. This is likewise to be regarded as only an introduction to the subject, and includes no discussions not capable of being given in a brief and easily intelligible form. In the body of the work, as well as in the Appendix, only so much is given either of principles or applications, as is thought to be desirable in a course of elementary instruction, after completing which, the student who desires to pursue the subject farther will be prepared to take up such advanced treatises as those of Chauvenet and Sawitsch.

TROY, N.Y., July, 1891.

CONTENTS.

CHAPTER I.

Definitions. — Spherical Problems.

CHAPTER II.

Conversion of Time. — Hour Angles.

CHAPTER III.

The Transit Instrument.

CHAPTER IV.

The Sextant.

CHAPTER V.

Finding the Time by Observation.

CHAPTER VI.

Finding Differences of Longitude.

CHAPTER VII.

Finding the Latitude of the Place.

CHAPTER VIII.

Finding the Azimuth of a Given Line.

CHAPTER IX.

Figure and Dimensions of the Earth.

APPENDIX.

The Method of Least Squares.

TABLES.

THE GREEK ALPHABET.

Letters.	Names.	Letters.	Names.
Α, α,	Alpha.	Ν, ν,	Nu.
Β, β,	Beta.	Ξ, ξ,	Xi.
Γ, γ,	Gamma.	Ο, ο,	Omicron.
Δ, δ,	Delta.	Π, π,	Pi.
Ε, ε,	Epsilon.	Ρ, ρ,	Rho.
Ζ, ζ,	Zeta.	Σ, σ,	Sigma.
Η, η,	Eta.	Τ, τ,	Tau.
Θ, θ,	Theta.	Υ, υ,	Upsilon.
Ι, ι,	Iota.	Φ, φ,	Phi.
Κ, κ,	Kappa.	Χ, χ,	Chi.
Λ, λ,	Lambda.	Ψ, ψ,	Psi.
Μ, μ,	Mu.	Ω, ω,	Omega.

CHAPTER I.

DEFINITIONS. — SPHERICAL PROBLEMS.

1. Spherical Astronomy. — This department of Astronomy arises from the application of Spherical Trigonometry to the Celestial Sphere. It takes no account of the real distances and magnitudes of the heavenly bodies, but only of their relative directions. Whatever their actual positions, they are all regarded as situated on the surface of a sphere of indefinitely great radius, of which the earth is the center.

2. Practical Astronomy. — This branch of the subject treats of the theory and use of astronomical instruments, and the practical solution of astronomical problems requiring data derived from observation.

Circles of the Celestial Sphere.

3. Axis and Poles. — The *axis* of the celestial sphere, PP', Fig. 1, is the earth's axis produced. Its extremities are the *poles* of the sphere.

4. Equator. — The *equator*, CWD, is the great circle whose plane is perpendicular to the axis.

5. Vertical Line. — The *vertical line*, ZON, is the line indicated by the direction of the plumb-line at the given place. It intersects the celestial sphere in the *zenith* and *nadir*.

6. Horizon. — The *horizon*, AWB, is the great circle whose plane is perpendicular to the vertical line. It has the zenith and nadir for poles.

7. **Meridian.** — The *meridian, APZB,* is the great circle whose plane passes through the zenith and the poles. It intersects the horizon in the *north* and *south* points, *A* and *B*.

8. **Prime Vertical.** — The *prime vertical, ZWNE,* is the great circle passing through the zenith perpendicular to the meridian. It intersects the horizon in the *east* and *west* points.

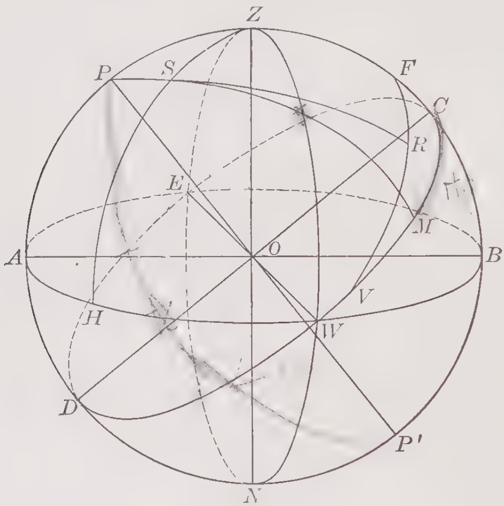

FIG. 1.

9. **Ecliptic and Equinoxes.** — The *ecliptic, VF,* is the great circle which the sun appears to describe during the year. The point *V,* where the sun crosses the equator from south to north, is the *vernal equinox.* The point where it crosses from north to south is the *autumnal equinox.*

The points of the ecliptic 90° distant from the equinoxes are called *solstices.*

10. **Obliquity.** — The *obliquity* of the ecliptic is the angle which its plane makes with that of the equator. It is about 23° 27′.

Spherical Co-ordinates.

11. The position of a body on the celestial sphere is determined by either of three systems of spherical co-ordinates.

In the first system the body's position is referred to the horizon, and the co-ordinates are called *azimuth* and *altitude*.

12. Vertical Circles. — Suppose a series of great circles drawn through the zenith and nadir, and, of course, perpendicular to the horizon. They are called *vertical circles.*

13. Azimuth. — The *azimuth* of a body is the arc of the horizon between the north point and the vertical circle passing through the body. Or, it is the angle at the zenith measured by this arc. If S, Fig. 1, is the position of a star, its azimuth is $AH = AZH$, and will be denoted by Z.

In a similar way, azimuth may be measured from the south point. The azimuth of the star S from the south point is $BH = BZH$, and denoted by Z'. We evidently have $Z' = 180° - Z$.

14. Altitude and Zenith Distance. — The *altitude* of a body is its distance from the horizon measured on a vertical circle. The altitude of the star S is HS, denoted by h.

The *zenith distance*, ZS, is the complement of the altitude. It is denoted by z, and we have $z = 90° - h$.

15. In the second system the body's position is referred to the equator, and the co-ordinates are called *right ascension* and *declination.*

16. Hour Circles. — Suppose a series of great circles drawn through the poles; that is, perpendicular to the equator. They are called *hour circles.*

The hour circle passing through the equinoxes is called the *equinoctial colure,* and that through the solstices, the *solstitial colure.*

17. Hour Angle. — The *hour angle* of a body, S, is the angle which its hour circle, PM, makes with the meridian; or the arc, CM, of the equator, which measures that angle. It is always measured westward from the meridian, and denoted by P.

18. Right Ascension. — The *right ascension* of a body is the distance on the equator from the vernal equinox, eastward, to the hour circle of the body. The right ascension of S is VM, denoted by α, or sometimes by R. A.

19. Declination and Polar Distance. — The *declination* of a body is its distance from the equator measured on an hour circle. If north of the equator, the declination is positive; if south, negative. The declination, MS, is denoted by δ.

The *polar distance,* PS, is denoted by p, and we have $p = 90° - \delta$.

20. In the third system, the place of the body is referred to the ecliptic, and the co-ordinates are called *celestial longitude* and *latitude.*

21. Latitude Circles. — Suppose a series of great circles drawn through the poles of the ecliptic: they are called *latitude circles.*

22. Celestial Longitude. — The *longitude* of a body is the distance, $VR = L$, on the ecliptic from the vernal equinox, eastward, to the latitude circle through the body.

23. Celestial Latitude. — The *latitude* of a body is its distance, $RS = l$, from the ecliptic, measured on a latitude circle.

24. The Nautical Almanac. — In general, the co-ordinates of the *second* system are those best adapted to use in the practical problems which will be considered in this work. Their values for any given date may be found from the Nautical Almanac, which contains the right ascension and declination of the sun, the moon, the major planets, and several hundred of the principal fixed stars; those of the moon being given for every hour, of the sun and planets for every day, and of the fixed stars for every ten days. As the values in the almanac are for given instants of time at the first meridian, it is necessary to know the observer's longitude in order to be able to find from the data in the almanac the values of the required co-ordinates for the time and place of observation.

Other tables contained in the Nautical Almanac will be referred to hereafter.

Co-ordinates of the Observer.

25. The position of the observer on the earth's surface is determined by the *longitude* and *latitude* of the place.

26. Longitude. — The *longitude of the place* is the arc of the equator intercepted between the meridian of the place and the first meridian. It is denoted by λ.

27. Latitude. — The *latitude of the place* is the declination of the zenith, $CZ = AP$, and is denoted by ϕ.

The *colatitude* is the complement of the latitude, $PZ = BC = \psi$.

Spherical Problems.

28. The Astronomical Triangle. — Many of the most important problems of Spherical Astronomy can be reduced to the solution of the spherical triangle PZS, Fig. 2, formed by joining the pole, the zenith, and the place of a star by arcs of great circles.

The three sides of this triangle are

$PZ = 90° - \phi$, the colatitude of the place.

$PS = 90° - \delta$, the polar distance of the star.

$ZS = 90° - h$, the zenith distance of the star;

and the three angles are

P, the star's hour angle.

Z, its azimuth from the north point.

S, which is called the parallactic angle.

29. Trigonometrical Formulæ. — The following well-known formulæ of Spherical Trigonometry, applied to the triangle PZS, will furnish most of the general equations required in the discussions which follow. Denoting the angles of any spherical triangle by A, B, C, and its sides by a, b, c, we have

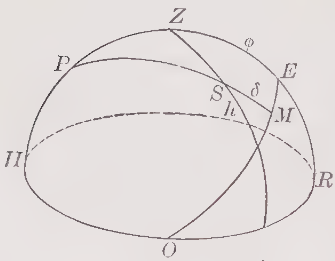

Fig. 2.

$$\left.\begin{aligned} \sin a \sin B &= \sin b \sin A \\ \sin b \sin C &= \sin c \sin B \\ \sin c \sin A &= \sin a \sin C \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \cos b &= \cos c \cos a + \sin c \sin a \cos B \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C \end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} \sin a \cos B &= \sin c \cos b - \cos c \sin b \cos A \\ \sin b \cos C &= \sin a \cos c - \cos a \sin c \cos B \\ \sin c \cos A &= \sin b \cos a - \cos b \sin a \cos C \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned}\sin^2 \tfrac{1}{2} A &= \frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}\\ \sin^2 \tfrac{1}{2} B &= \frac{\sin (s-c) \sin (s-a)}{\sin c \sin a}\\ \sin^2 \tfrac{1}{2} C &= \frac{\sin (s-a) \sin (s-b)}{\sin a \sin b}\end{aligned}\right\} \quad (4)$$

in which $s = \frac{1}{2} (a + b + c)$.

30. Astronomical Formulæ. — If we apply formulæ (1), (2), and (3) to the triangle PZS, making

$$\begin{aligned} A &= P, & a &= 90° - h,\\ B &= Z, & b &= 90° - \delta,\\ C &= S, & c &= 90° - \phi, \end{aligned}$$

we shall obtain

$$\cos h \sin Z = \cos \delta \sin P \quad (5)$$

$$\cos \delta \sin S = \cos \phi \sin Z \quad (6)$$

$$\cos \phi \sin P = \cos h \sin S \quad (7)$$

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos P \quad (8)$$

$$\sin \delta = \sin \phi \sin h + \cos \phi \cos h \cos Z \quad (9)$$

$$\sin \phi = \sin h \sin \delta + \cos h \cos \delta \cos S \quad (10)$$

$$\cos h \cos Z = \sin \delta \cos \phi - \cos \delta \sin \phi \cos P \quad (11)$$

$$\cos \delta \cos S = \sin \phi \cos h - \cos \phi \sin h \cos Z \quad (12)$$

$$\cos \phi \cos P = \sin h \cos \delta - \cos h \sin \delta \cos S \quad (13)$$

By making the proper substitutions in these equations we may find the formulæ for a body in any position in the heavens.

31. Problem. — Given the *latitude of the place* and the *declination of the body*, to find its *altitude* and *azimuth* when it is on the *six hour circle*.

In this position the hour angle $P = 6$ *hours* $= 90°$; hence $\sin P = 1$, $\cos P = 0$, and (8) becomes

$$\sin h = \sin \delta \sin \phi \quad (14)$$

(5) becomes $\cos h \sin Z = \cos \delta$;

(11) becomes $\cos h \cos Z = \sin \delta \cos \phi$;

whence by division,

$$\tan Z = \frac{\cot \delta}{\cos \phi} \tag{15}$$

Equations (14) and (15) are the expressions required.

32. Problem. — Given the same data as before, to find the *hour angle* and *azimuth* of a body in the *horizon.*

In this position, $h = 0°$, $\sin h = 0$, $\cos h = 1$, and, by (8),

$$\cos P = -\frac{\sin \delta \sin \phi}{\cos \delta \cos \phi} = -\tan \delta \tan \phi \tag{16}$$

and by (9),

$$\cos Z = \frac{\sin \delta}{\cos \phi} \tag{17}$$

33. Problem. — Given the same data, to find the *altitude* or *zenith distance* of a body on the *meridian.*

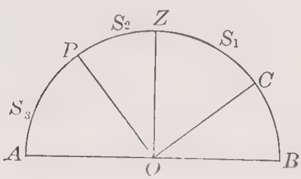

FIG. 3.

Let AZB, Fig. 3, be the meridian, Z the zenith, P the pole, and C a point of the equator. We have $CZ = \phi$, $PZ = \psi$, $ZS = z$, $CS = \delta$, and $PS = p$.

In respect to the position of a body on the meridian, there may be three cases. It may be

(1) South of the zenith, as at S_1. We shall then have

$$ZS_1 = ZC - CS_1, \quad \text{or} \quad z = \phi - \delta \tag{18}$$

(2) Between the zenith and the pole, as at S_2. Then

$$ZS_2 = CS_2 - CZ, \quad \text{or} \quad z = \delta - \phi \tag{19}$$

(3) Below the pole, as at S_3. Then

$$ZS_3 = ZP + PS_3, \text{ or } z = \psi + p = 180° - (\phi + \delta) \tag{20}$$

From these values of z we find at once

$$(1) \quad h = 90° - (\phi - \delta) = \psi + \delta \tag{21}$$

$$(2) \quad h = 90° - (\delta - \phi) = \phi + p \tag{22}$$

$$(3) \quad h = 90° - (\psi + p) = \phi - p \tag{23}$$

34. Problem. — Given the *latitude of the place*, and the *declination* and *zenith distance* of a body, to find its *hour angle* and *azimuth*.

Applying the first two of equations (4) to the triangle *PZS*, Fig. 2, making $a = z$ instead of $90° - h$, as before, we find

$$\sin^2 \tfrac{1}{2} P = \frac{\sin [s - (90° - \delta)] \sin [s - (90° - \phi)]}{\cos \delta \cos \phi} \qquad (24)$$

$$\sin^2 \tfrac{1}{2} Z = \frac{\sin [s - (90° - \phi)] \sin (s - z)}{\cos \phi \sin z} \qquad (25)$$

in which

$$s = \frac{z + (90° - \delta) + (90° - \phi)}{2} = 90° + \tfrac{1}{2}(z - \phi - \delta).$$

Hence

$$s - z = 90° + \tfrac{1}{2}(z - \phi - \delta) - z = 90° - \tfrac{1}{2}(z + \phi + \delta),$$
$$s - (90° - \phi) = \tfrac{1}{2}(z - \phi - \delta) + \phi = \tfrac{1}{2}(z + \phi - \delta),$$
$$s - (90° - \delta) = \tfrac{1}{2}(z - \phi - \delta) + \delta = \tfrac{1}{2}(z - \phi + \delta).$$

By the substitution of these values, (24) and (25) become

$$\sin^2 \tfrac{1}{2} P = \frac{\sin \tfrac{1}{2}(z - \phi + \delta) \sin \tfrac{1}{2}(z + \phi - \delta)}{\cos \delta \cos \phi} \qquad (26)$$

$$\sin^2 \tfrac{1}{2} Z = \frac{\sin \tfrac{1}{2}(z + \phi - \delta) \cos \tfrac{1}{2}(z + \phi + \delta)}{\cos \phi \sin z} \qquad (27)$$

the expressions required.

35. Problem. — Given the *latitude of the place*, and the *hour angle* and *declination* of a body, to find its *azimuth* and *altitude*.

Suppose the azimuth to be reckoned from the south point. We have $Z = 180° - Z'$, whence

$$\sin Z = \sin Z', \quad \cos Z = -\cos Z',$$

which values substituted in (5), (8), and (11), reduce them to

$$\cos h \sin Z' = \cos \delta \sin P \qquad (28)$$

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos P \qquad (29)$$

$$\cos h \cos Z' = -\sin \delta \cos \phi + \cos \delta \sin \phi \cos P \qquad (30)$$

Make $\sin\delta = m \sin M$ (31)

and $\cos\delta \cos P = m \cos M$ (32)

then (29) and (30) become

$$\sin h = m(\sin\phi \sin M + \cos\phi \cos M)$$
$$= m\cos(\phi - M) \qquad (33)$$
$$\cos h \cos Z' = m(\sin\phi \cos M - \cos\phi \sin M)$$
$$= m \sin(\phi - M) \qquad (34)$$

Dividing (31) by (32)

$$\tan M = \frac{\tan\delta}{\cos P} \qquad (35)$$

Dividing (28) by (34),

$$\tan Z' = \frac{\cos\delta}{m} \cdot \frac{\sin P}{\sin(\phi - M)};$$

but from (32),

$$\frac{\cos\delta}{m} = \frac{\cos M}{\cos P},$$

hence $$\tan Z' = \frac{\cos M \tan P}{\sin(\phi - M)} \qquad (36)$$

Dividing (34) by (33),

$$\frac{\cos Z'}{\tan h} = \tan(\phi - M),$$

whence $$\tan h = \frac{\cos Z'}{\tan(\phi - M)} \qquad (37)$$

Equations (35), (36), and (37) solve the problem.

36. To show that we are at liberty to make the assumptions expressed in (31) and (32), we observe that if we have any two real quantities, positive or negative, as x and y, we may put

$$x = m \sin M,$$
$$y = m \cos M;$$

as we then have

$$x^2 + y^2 = m^2(\sin^2 M + \cos^2 M) = m^2,$$

or $$m = \sqrt{x^2 + y^2};$$

and also $$\frac{x}{y} = \frac{\sin M}{\cos M} = \tan M,$$

or $$M = \tan^{-1}\frac{x}{y}.$$

These values of m and M are always real and possible, whatever be the values or signs of x and y; hence there are some real values of m and M which will satisfy (31) and (32).

37. Problem. — Given the *right ascension* and *declination* of a body, and the *obliquity of the ecliptic*, to find the *celestial longitude* and *latitude* of the body.

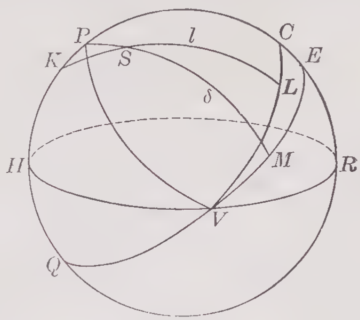

Fig. 4.

Let $HPER$, Fig. 4, be that position of the meridian which coincides with the solstitial colure, and which is therefore perpendicular to both the equator, EQ, and the ecliptic, CV. The vernal equinox V is then in the horizon, PV is the equinoctial colure, and the arcs CV, EV, and QV are quadrants. Let K be the pole of the ecliptic; then in the triangle KPS,

$$KP = CE = \omega = \text{obliquity of ecliptic};$$
$$PS = 90° - \delta; \quad KS = 90° - l;$$
$$\text{angle } KPS = \text{arc } QM = QV + VM = 90° + \alpha;$$
$$\text{angle } PKS = \text{arc } CL = CV - VL = 90° - L.$$

38. In the first equations of (1), (2), and (3), making $A = 90° + \alpha$, $B = 90° - L$, $a = 90° - l$, $b = 90° - \delta$, $c = \omega$, we obtain the following:

$$\cos l \cos L = \cos \delta \cos \alpha \tag{38}$$

$$\sin l = \sin \delta \cos \omega - \cos \delta \sin \omega \sin \alpha \tag{39}$$

$$\cos l \sin L = \sin \omega \sin \delta + \cos \omega \cos \delta \sin \alpha \tag{40}$$

In order to put these equations in a form adapted to logarithms, make

$$\sin \delta = m \sin M \tag{41}$$

$$\cos \delta \sin \alpha = m \cos M \tag{42}$$

which reduce (39) and (40) to

$$\begin{aligned} \sin l &= m (\sin M \cos \omega - \cos M \sin \omega) \\ &= m \sin (M - \omega) \end{aligned} \tag{43}$$

$$\begin{aligned} \cos l \sin L &= m (\cos M \cos \omega + \sin M \sin \omega) \\ &= m \cos (M - \omega) \end{aligned} \tag{44}$$

Dividing (41) by (42),

$$\tan M = \frac{\tan \delta}{\sin \alpha} \tag{45}$$

Dividing (44) by (38),

$$\tan L = \frac{m}{\cos \delta} \cdot \frac{\cos (M - \omega)}{\cos \alpha};$$

but by (42), $$\frac{m}{\cos \delta} = \frac{\sin \alpha}{\cos M},$$

whence $$\tan L = \frac{\tan \alpha \cos (M - \omega)}{\cos M} \tag{46}$$

Dividing (43) by (44),

$$\frac{\tan l}{\sin L} = \tan (M - \omega),$$

whence $$\tan l = \sin L \tan (M - \omega) \tag{47}$$

Equations (45), (46), and (47) solve the problem.

CHAPTER II.

CONVERSION OF TIME. — HOUR ANGLES.

Conversion of Time.

39. Sidereal and Solar Days. — A *sidereal* day is the interval between two successive meridian passages of the vernal equinox. An *apparent solar* day is the interval between two successive meridian passages of the sun's center. A *mean solar* day is the average length of all the apparent solar days in a tropical year.

40. Equation of Time. — The *equation of time* is a quantity which, being added to the apparent solar time, gives the mean solar time. Hence the mean time may be found from the apparent by adding the equation of time, that is, applying it according to its sign; and the apparent time may be found from the mean by applying the equation of time with a contrary sign.

The value of the equation of time for any day may be found from the *solar ephemeris,* or tables of the sun in the Nautical Almanac.

41. Mean Solar and Sidereal Intervals. — The tropical year contains, according to Bessel, 365.24222 mean solar days. But since the sun makes an apparent revolution from west to east in the ecliptic in a year, it loses one diurnal revolution from east to west, in comparison with the fixed stars; hence there are just one more sidereal than solar days in a year; namely. 366.24222.

We have then

$$365.24222 \textit{ solar days} = 366.24222 \textit{ sid. days},$$

or
$$1 \textit{ solar day} = 1.0027379 \textit{ sid. day} \tag{1}$$

and
$$1 \textit{ sid. day} = 0.9972696 \textit{ solar day} \tag{2}$$

which may be put in the form

$$1 \textit{ solar day} = 1 \textit{ sid. day} + 3^{m}\, 56^{s}.555 \textit{ sid. time};$$

$$1 \textit{ sid. day} = 1 \textit{ solar day} - 3^{m}\, 55^{s}.909 \textit{ solar time}.$$

The excess of a mean solar above a sidereal day is therefore $3^{m}\, 56^{s}.555$ of sidereal time, or $3^{m}\, 55^{s}.909$ mean solar time.

Since a sidereal day, hour, etc., is shorter than a solar day, hour, etc., a given interval will contain more sidereal than solar days or hours, etc. Hence it follows from (1) that any interval expressed in mean solar time may be changed into its sidereal equivalent by multiplying by 1.0027379; and from (2) that any sidereal interval may be changed into its mean solar equivalent by multiplying by 0.9972696.

If then we put

$$M = \text{any mean solar interval},$$

$$S = \text{equivalent sidereal interval};$$

then
$$S = 1.0027379\, M = (1 + 0.0027379)\, M,$$

$$M = 0.9972696\, S = (1 - 0.0027304)\, S.$$

Put
$$c = 0.0027379, \qquad c' = 0.0027304,$$

and we have

$$S = M + cM; \qquad M = S - c'S. \tag{3}$$

The values of cM and $c'S$ may be taken from the annexed tables, which are reduced from a larger table in the *American Ephemeris*.

Table *A*.

To find the sidereal equivalent of any interval of mean solar time.

$$S = M + cM.$$

Mean hours.	cM	Mean min.	cM	Mean min.	cM	Mean sec.	cM	Mean sec.	cM
	m s		s		s		s		s
1	0 9.856	1	0.164	31	5.093	1	0.003	31	0.085
2	0 19.713	2	0.329	32	5.257	2	0.005	32	0.088
3	0 29.569	3	0.493	33	5.421	3	0.008	33	0.090
4	0 39.426	4	0.657	34	5.585	4	0.011	34	0.093
5	0 49.282	5	0.821	35	5.750	5	0.014	35	0.096
6	0 59.139	6	0.986	36	5.914	6	0.016	36	0.099
7	1 8.995	7	1.150	37	6.078	7	0.019	37	0.101
8	1 18.852	8	1.314	38	6.242	8	0.022	38	0.104
9	1 28.708	9	1.478	39	6.407	9	0.025	39	0.107
10	1 38.565	10	1.643	40	6.571	10	0.027	40	0.110
11	1 48.421	11	1.807	41	6.735	11	0.030	41	0.112
12	1 58.278	12	1.971	42	6.900	12	0.033	42	0.115
13	2 8.134	13	2.136	43	7.064	13	0.036	43	0.118
14	2 17.991	14	2.300	44	7.228	14	0.038	44	0.120
15	2 27.847	15	2.464	45	7.392	15	0.041	45	0.123
16	2 37.704	16	2.628	46	7.557	16	0.044	46	0.126
17	2 47.560	17	2.793	47	7.721	17	0.047	47	0.129
18	2 57.417	18	2.957	48	7.885	18	0.049	48	0.131
19	3 7.273	19	3.121	49	8.049	19	0.052	49	0.134
20	3 17.129	20	3.285	50	8.214	20	0.055	50	0.137
21	3 26.986	21	3.450	51	8.378	21	0.057	51	0.140
22	3 36.842	22	3.614	52	8.542	22	0.060	52	0.142
23	3 46.699	23	3.778	53	8.707	23	0.063	53	0.145
24	3 56.555	24	3.943	54	8.871	24	0.066	54	0.148
		25	4.107	55	9.035	25	0.068	55	0.151
		26	4.271	56	9.199	26	0.071	56	0.153
		27	4.435	57	9.364	27	0.074	57	0.156
		28	4.600	58	9.528	28	0.077	58	0.159
		29	4.764	59	9.692	29	0.079	59	0.162
		30	4.928	60	9.856	30	0.082	60	0.164

Table *B*.

To find the mean equivalent of any interval of sidereal time.

$$M = S - c'S.$$

Sid. hours.	$c'S$	Sid. min.	$c'S$	Sid. min.	$c'S$	Sid. sec.	$c'S$	Sid. sec.	$c'S$
	m s		s		s		s		s
1	0 9.830	1	0.164	31	5.079	1	0.003	31	0.085
2	0 19.659	2	0.328	32	5.242	2	0.005	32	0.087
3	0 29.489	3	0.491	33	5.406	3	0.008	33	0.090
4	0 39.318	4	0.655	34	5.570	4	0.011	34	0.093
5	0 49.148	5	0.819	35	5.734	5	0.014	35	0.096
6	0 58.977	6	0.983	36	5.898	6	0.016	36	0.098
7	1 8.807	7	1.147	37	6.062	7	0.019	37	0.101
8	1 18.636	8	1.311	38	6.225	8	0.022	38	0.104
9	1 28.466	9	1.474	39	6.389	9	0.025	39	0.106
10	1 38.296	10	1.638	40	6.553	10	0.027	40	0.109
11	1 48.125	11	1.802	41	6.717	11	0.030	41	0.112
12	1 57.955	12	1.966	42	6.881	12	0.033	42	0.115
13	2 7.784	13	2.130	43	7.045	13	0.035	43	0.117
14	2 17.614	14	2.294	44	7.208	14	0.038	44	0.120
15	2 27.443	15	2.457	45	7.372	15	0.041	45	0.123
16	2 37.273	16	2.621	46	7.536	16	0.044	46	0.126
17	2 47.102	17	2.785	47	7.700	17	0.046	47	0.128
18	2 56.932	18	2.949	48	7.864	18	0.049	48	0.131
19	3 6.762	19	3.113	49	8.027	19	0.052	49	0.134
20	3 16.591	20	3.277	50	8.191	20	0.055	50	0.137
21	3 26.421	21	3.440	51	8.355	21	0.057	51	0.139
22	3 36.250	22	3.604	52	8.519	22	0.060	52	0.142
23	3 46.080	23	3.768	53	8.683	23	0.063	53	0.145
24	3 55.909	24	3.932	54	8.847	24	0.066	54	0.147
		25	4.096	55	9.010	25	0.068	55	0.150
		26	4.259	56	9.174	26	0.071	56	0.153
		27	4.423	57	9.338	27	0.074	57	0.156
		28	4.587	58	9.502	28	0.076	58	0.158
		29	4.751	59	9.666	29	0.079	59	0.161
		30	4.915	60	9.830	30	0.082	60	0.164

42. Examples.—A few examples of the conversion of mean solar and sidereal intervals are here given as a practical application of equations (3).

1. Given $M = 9^h\,44^m\,38^s.66$, to find S.

$$\text{Table } A. \begin{cases} \text{For } 9^h, & 1^m\,28^s.708 \\ \quad 44^m, & 7\,.228 \\ \quad 38\tfrac{2}{3}^s, & 0\,.106 \end{cases}$$

$$\begin{aligned} cM &= 1\ \ 36\,.042 \\ M &= 9\ \ 44\ \ 38\,.660 \\ S = M + cM &= 9^h\,46^m\,14^s.702 \end{aligned}$$

2. Given $S = 9^h\,46^m\,14^s.702$, to find M.

$$\text{Table } B. \begin{cases} \text{For } 9^h, & 1^m\,28^s.466 \\ \quad 46^m, & 7\,.536 \\ \quad 14^s.7, & 0\,.040 \end{cases}$$

$$\begin{aligned} c'S &= 1\ \ 36\,.042 \\ S &= 9\ \ 46\ \ 14\,.702 \\ M = S - c'S &= 9^h\,44^m\,38^s.66 \end{aligned}$$

3. Given $M = 8^h\,50^m\,05^s.09$, to find S.
4. Given $M = 6^h\,34^m\,41^s.68$, to find S.
5. Given $M = 13^h\,17^m\,10^s.73$, to find S.
6. Given $S = 8^h\,51^m\,32^s.17$, to find M.
7. Given $S = 6^h\,35^m\,46^s.52$, to find M.
8. Given $S = 13^h\,19^m\,21^s.69$, to find M.

43. Mean and Sidereal Time at any Instant.—Let

$m =$ mean time at given instant,
$s =$ sidereal time at given instant,
$s' =$ sidereal time at preceding mean noon.

Then $m =$ mean interval elapsed since mean noon,
and $s - s' =$ sidereal interval elapsed since mean noon.

Hence $$m = \textit{mean equivalent of } (s - s') \tag{4}$$
and $$s - s' = \textit{sidereal equivalent of } (m) \tag{5}$$

If then the sidereal time at any instant is given, the mean time at the same instant is found by (4), which gives the rule:

From the given sidereal time subtract the sidereal time at preceding mean noon, and reduce the remainder to its mean equivalent.

Conversely, if the mean time at any instant is given to find the sidereal time, we have from (5),

$$s = s' + \textit{sidereal equivalent of } (m) \tag{6}$$

whence the rule:

To the sidereal time at preceding mean noon add the given mean time reduced to its sidereal equivalent.

The "sidereal time at mean noon" is given for every day in the solar ephemeris.

Examples.

1\. Given $\left\{\begin{array}{l} s\ = 19^h\,45^m\ \ 2^s.05 \\ s' = 13\ \ 01\ \ \ 56\,.52 \end{array}\right\}$ to find m.

$$s - s' = \ \ 6\ \ 43\ \ \ 05\,.53$$

This sidereal interval is to be reduced to its mean equivalent by Table B.

	We find, for 6^h, $58^s.977$
$s - s' = 6^h\,43^m\,05^s.53$	43^m, $7\,.045$
$c'S = \ \ \ 1\ \ 06\,.04$	$5^s.53$, $.015$
$m = 6\ \ 41\ \ \ 59\,.49$	$c'S = 66.04$

2\. Given $\left\{\begin{array}{l} m = \ \ 5^h\,26^m\,03^s.32 \\ s' = 23\ \ 02\ \ \ 10\,.34 \end{array}\right\}$ to find s.

Table A, $cM = \qquad\qquad 53\,.56$

$$s = \ \ 4\ \ 29\ \ \ 07\,.22$$

3. Given $s = 14^h\,10^m\,21^s.39$, $s' = 9^h\,7^m\,24^s.89$, to find m.

4. Given $s = 16^h\,25^m\,13^s.79$, $s' = 10^h\,38^m\,5^s.62$, to find m.

5. Given $m = 6^h\,53^m\,47^s.24$, $s' = 10^h\,14^m\,26^s.31$, to find s.

6. Given $m = 7^h\,27^m\,33^s.97$, $s' = 5^h\,50^m\,17^s.08$, to find s.

7. April 19, the sidereal and mean solar clocks were compared, with the following result, viz.:

sid. clock, $7^h\,47^m\,22^s.5$; mean clock, $5^h\,55^m\,00^s.0$;

the sidereal clock was found by observation to be $25^s.09$ slow, and $s' = 1^h\,51^m\,54^s.80$. Find the error of the mean solar clock. *Ans.* $5^s.51$ fast.

8. August 20, the comparison showed

sid. clock, $10^h\,27^m\,21^s$; mean solar clock, $00^h\,30^m\,30^s$;

the mean time clock was $6^s.46$ fast, and $s' = 9^h\,56^m\,51^s.21$. Find the error of the sidereal clock. *Ans.* $1^s.26$ fast.

9. Find the mean time of meridian passage of ζ *Virginis,* June 15, having given

right ascension $= 13^h\,28^m\,46^s.59$; $s' = 5^h\,34^m\,30^s.85$.

Ans. $m = 7^h\,52^m\,58^s.04$.

10. Find the mean time of meridian passage of η *Ursa Majoris,* June 19, having given

right ascension $= 13^h\,42^m\,57^s.86$; $s' = 5^h\,50^m\,17^s.08$.

Ans. $m = 7^h\,51^m\,23^s.34$.

11. Find the mean time of meridian passage of γ *Aquilœ.* Nov. 9, having given

right ascension $= 19^h\,40^m\,54^s.38$; $s' = 15^h\,14^m\,10^s.36$.

Ans. $m = 4^h\,26^m\,0^s.32$.

12. Find the mean time of meridian passage of θ *Aquilœ,* Nov. 9, having given

right ascension $= 20^h\,05^m\,29^s.80$; $s' = 15^h\,14^m\,10^s.36$.

Ans. $m = 4^h\,50^m\,31^s.71$.

44. Time and Arc. — From the fundamental relation

$$360° = 24^{h}, \qquad \text{or} \qquad 90° = 6^{h},$$

we have

$$\begin{aligned} 15° &= 1^{h}, & 1° &= 4^{m}, \\ 15' &= 1^{m}, & 1' &= 4^{s}, \\ 15'' &= 1^{s}, & 1'' &= 0^{s}.0666.... \end{aligned}$$

By means of this table arcs of the equator, such as hour angles, right ascensions, or terrestrial longitudes, may be changed from degrees, minutes, and seconds, into hours, minutes, and seconds, or *vice versa.*

EXAMPLES.

1. Express 29° 59′ 22″.125 in hours, minutes, and seconds.

Since

$$\begin{array}{lll} 15° = 1^{h}, & 1° = 4^{m}, & 29° = 15° + 14° = 1^{h}\,56^{m}\,00^{s}. \\ 15' = 1^{m}, & 1' = 4^{s}, & 59' = 45' + 14' = \quad 3\ \ 56\ . \\ 15'' = 1^{s}, & 1'' = 0^{s}.0666; & 22'' = 15'' + \ 7'' = \qquad 1\ .467 \\ & & 0''.125 = \frac{1}{8}(0.066) = \qquad 0\ .008 \\ \hline & & \qquad 1^{h}\,59^{m}\,57^{s}.475 \end{array}$$

2. Express $1^{h}\,59^{m}\,57^{s}.475$ in degrees, minutes, and seconds.

We find

$$\begin{array}{rl} 1^{h} = & 15°\ 00'\ 00''. \\ 59^{m} = 56^{m} + \ 3^{m} = & 14\ \ 45\ \ 00. \\ 57^{s} = 56^{s} + \ 1^{s} = & \quad\ \ 14\ \ 15. \\ 0.475 \times 15 = & \qquad\quad 7.\ 125 \\ \hline & 29°\ 59'\ 22''.125 \end{array}$$

3. Express 118° 11′ 38″ in hours, minutes, and seconds.
4. Express $7^{h}\,52^{m}\,46^{s}.533$ in degrees, minutes, and seconds.
5. Express 27° 41′ 42″ in hours, minutes, and seconds.
6. Express $1^{h}\,50^{m}\,46^{s}.8$ in degrees, minutes, and seconds.

Hour Angles.

45. The Sun's Hour Angle. — Since the apparent solar day begins when the sun's center is on the meridian, it follows that *the apparent time at a given instant is equal to the sun's hour angle.*

Hence, if the sun's hour angle be given, to find the mean solar time, we have, Art. 40,

$$\textit{mean time} = \textit{sun's hour angle} + \textit{equation of time} \qquad (7)$$

and if the mean time be given, to find the sun's hour angle,

$$\textit{sun's hour angle} = \textit{mean time} - \textit{equation of time} \qquad (8)$$

in which, of course, the sun's hour angle must be expressed in *time.*

46. Hour Angle of a Star. — Since the sidereal day begins when the vernal equinox is on the meridian, it follows that *the sidereal time at a given instant is equal to the hour angle of the vernal equinox.*

If, then, the hour angle of a star be given, to find the sidereal time, we have, Fig. 1, $CV = CM + MV$; that is,

$$\textit{sidereal time} = \textit{star's hour angle} + \textit{star's R. A.} \qquad (9)$$

in which, if the sum of the hour angle and right ascension should be greater than 24 hours, the excess will be the sidereal time required.

Conversely, if the sidereal time at any instant be given, to find the hour angle of a star, we have

$$\textit{star's hour angle} = \textit{sidereal time} - \textit{star's R. A.} \qquad (10)$$

and if the sidereal time be less than the right ascension, it must be increased by 24 hours, to render the subtraction possible.

In applying (9) and (10), the hour angle, as well as right ascension, must be expressed in *time.*

47. Hour Angles found by Observation. — Since a star is regarded as a fixed point, and the diurnal motions are uniform, it follows that equal *altitudes* of a star correspond to equal *hour angles.* If, then, observations be made on a star at equal altitudes east and west of the meridian, and if t and t' be the observed times of equal altitude, we shall have

$$\frac{t'-t}{2} = \text{star's hour angle at either observation,} \qquad (11)$$

$$\frac{t+t'}{2} = \text{observed time of meridian passage.} \qquad (12)$$

The practical advantage of equation (11) is that whatever may be the error of the clock, it is eliminated in taking the difference of observed times. This, however, supposes the error to be constant between the observations.

CHAPTER III.

THE TRANSIT INSTRUMENT.

Description, Adjustment, and Use of the Transit Instrument.

48. Description of the Transit Instrument. — The Transit Instrument is an instrument designed to facilitate the observation of meridian passages of the heavenly bodies. It consists essentially of a telescope supported on, and capable of motion about, an axis perpendicular to itself, and which can also be made perpendicular to the meridian plane. In the *portable* transit, one form of which is represented in Fig. 5, the axis is supported by a framework of cast iron, which rests on foot-plates secured to the pier cap. The pivots of the axis rest in Y-supports, and for adjusting it accurately in position, one of these supports admits of a small motion in azimuth by means of a fine-motion screw with a micrometer head, while the other may, by similar means, be moved a little in altitude.

The tube which carries the eye-piece, and which slides in and out of the main telescope tube by a rack and pinion, carries also a diaphragm containing the *reticle*, Fig. 6, which consists of a system of parallel transit lines, and one or two others at right angles to them: the diaphragm is capable of being revolved so that the transit lines can be made, and clamped, truly vertical.

The lines may be either spider lines or fine platinum wire, or, better still, may be ruled on glass; but of whatever kind, they are, for convenience, called *wires*. The transit wires are uneven in number, one being placed in the

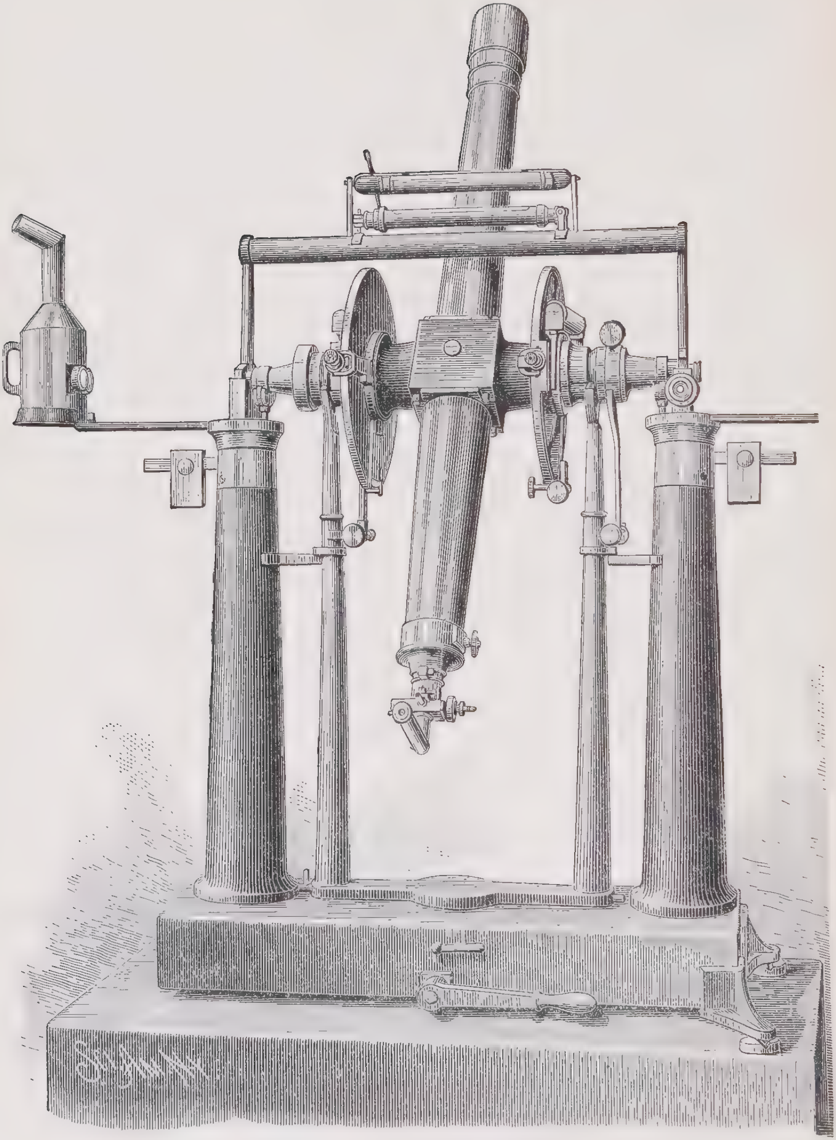

Fig. 5. — The Transit Instrument.

(By Fauth & Co., Washington, D C.)

middle and an equal number symmetrically arranged on each side. The diaphragm is also movable to the right or left by a screw, for the purpose of adjusting the vertical wires in position.

The middle vertical wire constitutes a visible artificial meridian, and the various appliances of the instrument furnish the means for making it coincide approximately with the true meridian. The other vertical wires are added for the purpose of multiplying observations, and thus securing greater accuracy by using the mean of several instead of a single one.

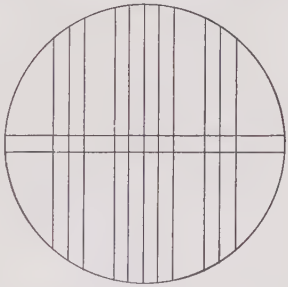

FIG. 6.

In order that the wires may be visible at night, it is necessary to illuminate the field of view. For this purpose the axis is made hollow, and a small reflector is placed diagonally within the telescope tube, so that the light of a lamp entering at one end of the axis will be reflected down upon the field of view.

The instrument is provided with one, and sometimes two, finding circles with levels attached for setting it on a given star. They may be set so as to read either altitudes or zenith distances.

Instruments of three inches or more aperture are also provided with apparatus by means of which the axis may be easily and quickly lifted out of its supports, and reversed end for end; a process often necessary in the practical use of the instrument.

49. The Clock. — Although the Transit Instrument is the most important instrument of Practical Astronomy, we can measure nothing with it alone, but only observe the passage

of bodies over the meridian. Its indispensable adjunct is a time-keeper, since the object of the observation is always to find the *time* of meridian passage. Whether a clock, chronometer, or common watch be used for this purpose, we shall, for brevity, call it simply the "clock." For observations on the fixed stars it will be convenient to have it regulated to sidereal time, but for sun transits, either sidereal or mean solar time may be used.

The clock "error" or "correction" is found by observing the transit of a star whose right ascension is given. If t denote the observed sidereal time of transit of a star whose right ascension is α, the clock correction is $\alpha - t$.

Having determined the correction of the sidereal clock, the right ascension of a star or other body, if unknown, will be found by simply observing the sidereal time of its meridian passage.

50. Adjustments of the Transit Instrument. — In order that the wires may be distinctly visible to different observers, their distance from the eye-piece must be capable of being slightly changed, and for this purpose the eye-piece can be moved a little back and forth in its tube until the wires appear distinct. The wires are then placed in the principal focus of the objective by sliding the tube carrying the wires and eye-piece in the main tube until the image of a star appears distinct. The adjustment may be verified by placing one of the horizontal wires on the star, and noticing whether the star remains on the wire when the eye is moved a little up and down, as it should, of course, if the wire is at the focus.

To set the transit wires vertical, adjust the telescope on some small distant object, and bisect it with the middle wire. It should remain on the wire as the telescope is slightly elevated or depressed, and if it does not, the diaphragm must be revolved a little. until this test is satisfied.

The right line, from the optical center of the objective perpendicular to the axis of rotation of the telescope, is called the *collimation axis.* The adjustment of the transit consists in placing the middle vertical wire in the collimation axis and the axis of rotation horizontal, and in the plane of the prime vertical. Hence, the three principal adjustments are those for *collimation*, *level*, and *azimuth.*

51. Collimation Adjustment. — To place the middle wire in the collimation axis, adjust the telescope on the distant point as before, then lift it from its bearings, and shift the axis end for end. Now point to the distant object, and if it is still bisected by the middle wire, the adjustment is complete; if not, correct half the error by moving the diaphragm a little to the right or left, place the wire again on the mark by the azimuth screw of the Y; repeat the test by reversing the axis again, and so on, till it is satisfied.

52. Level Adjustment. — To make the axis horizontal, apply the striding level to the pivots, note the readings of both ends of the bubble, then reverse the level and read again; the readings in the second position should be the reverse of those in the first. If they are not, one end of the axis must be raised or lowered by the adjusting screw of the Y until the test is satisfied.

53. Azimuth Adjustment. — This adjustment consists of three successive steps:

(1) Having a sidereal clock approximately correct, point the transit telescope to a close circumpolar star at the moment when its right ascension is indicated by the clock. This is done by moving the framework of the instrument so as to make the middle wire bisect the star. The instrument is now nearly in the meridian, but a closer approximation can be made after finding the error of the clock.

(2) As the instrument is nearly in the meridian, if it be carefully levelled, the telescope by its rotation will describe a vertical circle which, near the zenith, will almost coincide with the meridian; hence by observing the transit of one or two stars near the zenith, the error of the clock will be found with sufficient accuracy.

(3) Now turn down the telescope upon another star about to pass the meridian near the pole. The star will be not far from the middle wire, and moving very slowly. By the micrometer screw for azimuth motion place the middle wire on the star, and follow it till it reaches the meridian, the clock time of meridian passage being ascertained by applying the error of the clock to the right ascension of the star. If necessary, this last operation can be repeated on another circumpolar star.

The direction of the meridian line having been determined, a distant meridian mark may be set up, and the azimuth adjustment will afterwards consist in placing the middle wire on the mark, or turning it off, by the azimuth screw, the known distance of the mark from the meridian line.

54. Collimator. — If the transit instrument is permanently mounted on a pier, a collimator mounted on another pier near the transit may be used instead of a distant mark. A collimator is a telescope with cross-wires in its principal focus, mounted horizontally either north or south of the transit instrument, so that when the latter is turned into a horizontal position, the observer may look through it into the object-glass of the collimator. The rays of light from the cross-wires emerge from the object-glass in parallel lines, hence the wires appear as they would if viewed from an infinite distance.

55. Value of a Revolution of the Azimuth Screw. — In order to be able to move the instrument through a given

amount in azimuth by the micrometer screw provided for that purpose, the value of a revolution of this screw in seconds must be determined. This may be done by computing, by the formulæ of Art. 71, the azimuth deviation for two positions which differ by a single revolution, or by a known number of revolutions or parts of a revolution. The difference of the results obtained for the two positions, divided by the number of revolutions and multiplied by 15, will be the value of one revolution in seconds of arc. The head of the screw being divided into 100 equal parts, the value of any fraction of a revolution is also known.

56. Equatorial Intervals. — The time required for a star situated in the equator to pass the interval between two of the vertical wires is called their *equatorial interval.*

Suppose we observe the transit of a star whose declination is δ, note the exact time of its passing each of the vertical wires, and find the intervals; these intervals will be greater as δ is greater. For moderate declinations, two consecutive wires may be considered to intercept similar arcs of the diurnal circles; but for any two circles, similar arcs are proportional to their radii, that is, to the cosines of their declinations, and the diurnal velocities in these two circles are evidently in the same ratio. But the times required by two stars to pass the interval between the wires are inversely as the velocities; hence if we let

$I' =$ observed interval for star whose declination is δ,

$I =$ corresponding equatorial interval;

then $$I' : I : : 1 : \cos\delta,$$

or $$I = I' \cos\delta \tag{1}$$

57. The Mean Wire. — It is generally more convenient to find the equatorial interval of each wire from the mean wire. The *mean wire* is an imaginary wire so situated that

the time of transit over it is the mean of the times of transit over all the vertical wires.

To find the equatorial interval of each wire from the mean wire, observe the times of passage of a star over all the wires and take their mean; subtract the time for each wire from this mean and multiply the remainders by $\cos \delta$.

It will be best to use stars of considerable declination for this purpose, since errors in the observed times will be reduced in the ratio of 1 to $\cos \delta$. Of course, a large number of observations is requisite in order to obtain the results with much accuracy.

It should be observed that the equatorial intervals of wires situated on opposite sides of the middle one have contrary signs.

58. Use of the Equatorial Intervals. — A knowledge of the equatorial intervals is necessary to enable us to reduce incomplete observations; that is, those in which we have failed for any reason to observe a transit over a part of the wires. In such a case, to find the time of transit over the mean wire, we take the mean of the observed times, and apply to it as a correction the mean of the equatorial intervals of the wires observed, multiplied by $\sec \delta$, since from equation (1) we find

$$I' = I \sec \delta \tag{2}$$

59. Personal Equation. — Owing to peculiar habits acquired by every observer, it is found that even the most skilful and experienced observers differ by an appreciable and nearly constant quantity in their observation of the times of passage of a star over the transit wires. This difference is called their *personal equation*, and should be known and applied whenever the observations of two or more persons are combined together.

One way of finding the personal equation is as follows:

Let one observer note the times of transit of a star over the wires on one side of the middle one, and the other observer over the wires on the other side, and let each set of observations be reduced to the mean wire by means of the equatorial intervals; the difference of the two results will be their *relative* personal equation. An accurate determination, however, would require a large number of such observations.

60. Method of making Transit Observations. — A star is said to *culminate* when it passes the meridian. The time of culmination is observed by means of a transit instrument and sidereal clock in the following manner:

The right ascension of the star indicates the sidereal time of its culmination. Shortly before this time, the transit being placed in adjustment, the telescope is set and clamped at the zenith distance of the star on the meridian, previously determined. A few seconds before the time of transit the star will appear in the field of view, and will apparently move slowly across the field; and the object of the observer is to note the exact time by the clock when it crosses each vertical wire. As the passage of a wire will seldom occur simultaneously with the beat of the clock, the interval between them is to be estimated to the tenth of a second.

The observed time of passing each wire being thus noted, the mean of the observations on all the wires is taken as the observed time of culmination of the star. Examples of the reduction of transit observations over five and eleven wires respectively are given in Chapter V.

In observing transits of the sun, both the east and west limbs are observed, and the observer notes the instant at which the edge of the disc is tangent to each vertical wire. The observations of each limb are then reduced separately in the same way as those of a star, and the mean of the results is the observed time of culmination of the sun's center.

61. The Chronograph. — The method of observing transits explained above is called the "eye and ear" method; a more convenient, as well as more exact method of recording transit observations is by means of the electro-chronograph.

The object of the chronograph is to register transit or other observations by electricity. It consists, Fig. 7, of a cylinder which, by means of clock-work regulated by a governor, is made to rotate on its axis uniformly once in a

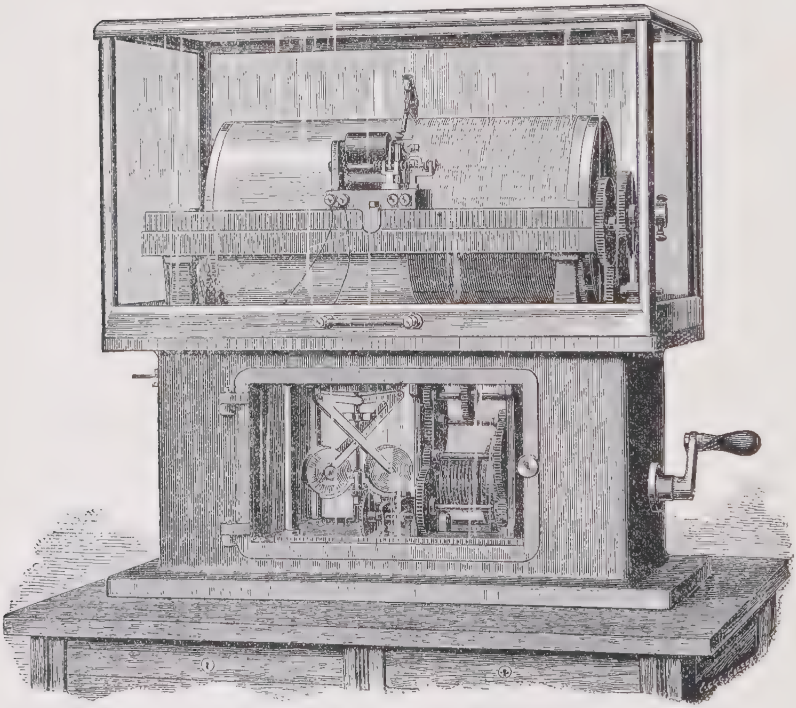

Fig. 7. — The Chronograph.

(By Warner & Swasey, Cleveland, O.)

minute. A sheet of paper is wrapped around the cylinder, and a pen is attached to a carriage, which, as the cylinder rotates, is carried slowly along parallel to its axis. If the pen is allowed to touch the paper, it will trace a spiral line, which, when the paper is removed from the cylinder, will appear as a series of parallel straight lines, about one-twelfth of an inch apart.

An electric current is made to pass from a battery through the clock, through an electro-magnet attached to the pen-carriage, and through a key in the hand of the observer, and it is so arranged that the circuit is made or broken for an instant by the clock at every second. This causes the electro-magnet to move the pen a little to one side, and thus the lines marked by the pen are broken by a series of equidistant notches representing seconds.

By pressing his key at the instant when a star crosses each wire in the field of view, the observer causes the pen to make a notch corresponding to each wire, and thus his observation is permanently registered. By marking some hour, minute, and second at its place on the paper, he can, at any time afterwards, find the minute and second at which the star crossed each wire, and the fraction of a second may be accurately measured by a scale.

62. Reversal of the Axis. — A transit observation can be made with the rotation axis of the instrument in either position. If the collimation adjustment is not perfect, the sign of the error is changed by reversing the axis; hence when several stars are observed for the purpose of finding the time, one half the observations should be made with the axis in one position and the other half with the axis reversed. This will eliminate the error of the collimation adjustment, and any error arising from inequality of the pivots.

Correction of Transit Observations.

63. Effect of Errors of Adjustment. — When the transit instrument is perfectly adjusted, the middle wire is in the collimation axis, and the axis of rotation is horizontal, and in the plane of the prime vertical. Now, in practice, these conditions are never exactly fulfilled, and when great accuracy is desired, it becomes necessary to apply to the times of observed transits, corrections required by the small errors of adjustment. These corrections we now proceed to determine.

Let BZD, Fig. 8, be the meridian, BAD the horizon, A the east point of the horizon, O the point where the rotation axis produced intersects the celestial sphere, and S the position of a star when observed on the middle wire. The point O would coincide with A, were it not for errors in the *level* and *azimuth* adjustments; the result is that the middle wire is thrown a little to one side of the meridian, and the *collimation* error produces a similar effect. Let each of these errors be considered *positive* when its effect is to throw the middle wire *east* of the meridian, the instrument being supposed to point south.

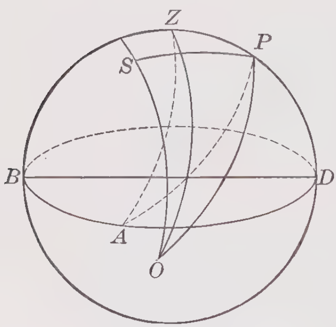

Fig. 8.

64. Investigation of the Corrections. — The hour angle of the point S is the correction to be made in the observed time of transit on account of the errors of adjustment.

Let $t = ZPS =$ hour angle of point S,

$T =$ observed sidereal time of transit,

$$\begin{aligned} \alpha &= \text{sidereal time of meridian transit,} \\ &= \text{star's right ascension,} \\ \Delta T &= \text{clock error;} \end{aligned}$$

then will

$$\alpha = T + \Delta T + t \tag{3}$$

To find t we may proceed as follows:

Comparing the points A and O, let their difference of hour angle, azimuth, zenith distance, and polar distance be denoted by e, a, b, and d respectively; then will

$$\begin{aligned} e &= ZPO - ZPA = ZPO - 90^\circ, \\ a &= PZA - PZO = 90^\circ - PZO, \\ b &= ZO - ZA = ZO - 90^\circ, \\ d &= PO - PA = PO - 90^\circ. \end{aligned}$$

Let also

$$c = \text{distance of middle wire from collimation axis.}$$

Now we have in the triangle PZO, $ZPO = 90^\circ + e$, $PZO = 90^\circ - a$, $ZO = 90^\circ + b$, $PO = 90^\circ + d$, $PZ = 90^\circ - \phi$; and in the triangle PSO,

$$SPO = ZPO - ZPS = (90^\circ + e) - t = 90^\circ - (t - e),$$

$$SO = 90^\circ - c \quad \text{and} \quad PS = 90^\circ - \delta.$$

65. The triangle PSO gives, by (2) of Chapter I.,

$$\cos SO = \cos PS \cos PO + \sin PS \sin PO \cos SPO,$$

or

$$\sin c = -\sin\delta \sin d + \cos\delta \cos d \sin(t - e) \tag{4}$$

The triangle PZO gives, by (2) and (3) of Chapter I.,

$$\cos PO = \cos PZ \cos ZO + \sin PZ \sin ZO \cos PZO,$$

$$\sin PO \cos ZPO = \sin PZ \cos ZO - \cos PZ \sin ZO \cos PZO;$$

or

$$\sin d = \sin\phi \sin b - \cos\phi \cos b \sin a \tag{5}$$

$$\cos d \sin e = \cos\phi \sin b + \sin\phi \cos b \sin a \tag{6}$$

66. Bessel's Formula. — As the instrument is supposed to be in approximate adjustment, the arcs a, b, c, d, e, and t are very small, and we may put for their sines the arcs themselves, and for their cosines unity. Equations (4), (5), and (6) thus reduce to

$$c = -d \sin \delta + (t - e) \cos \delta \qquad (7)$$

$$d = b \sin \phi - a \cos \phi \qquad (8)$$

$$e = b \cos \phi + a \sin \phi \qquad (9)$$

From (7) we find at once

$$t = e + d \tan \delta + c \sec \delta \qquad (10)$$

which is *Bessel's* formula for t.

67. Hansen's Formula. — Another formula for t is obtained as follows:

Multiplying (8) by $\sin \phi$, and (9) by $\cos \phi$, and adding, gives

$$b = d \sin \phi + e \cos \phi,$$

whence $$e = b \sec \phi - d \tan \phi,$$

which, substituted in (10), gives

$$t = b \sec \phi + d (\tan \delta - \tan \phi) + c \sec \delta \qquad (11)$$

which is *Hansen's* formula.

68. Mayer's Formula. — Still another expression for t is found thus:

Substitute in (10) the values of d and e from (8) and (9), and we have

$$t = b \cos \phi + a \sin \phi + b \sin \phi \tan \delta - a \cos \phi \tan \delta + c \sec \delta$$

$$= \frac{b \cos \phi \cos \delta + a \sin \phi \cos \delta + b \sin \phi \sin \delta - a \cos \phi \sin \delta + c}{\cos \delta}$$

$$= a \frac{\sin (\phi - \delta)}{\cos \delta} + b \frac{\cos (\phi - \delta)}{\cos \delta} + c \frac{1}{\cos \delta} \qquad (12)$$

which is *Mayer's* formula. If we put

$$\frac{\sin(\phi-\delta)}{\cos\delta}=A,\ \frac{\cos(\phi-\delta)}{\cos\delta}=B,\ \frac{1}{\cos\delta}=C,$$

it takes the simple form

$$t=aA+bB+cC \tag{13}$$

69. Best Position for Time Stars. — For a star in the zenith, $\delta=\phi$, and (11) and (12) both reduce to

$$t=(b+c)\sec\delta.$$

Hence, for such a star, the azimuth error has no effect on the observed time of transit, and t depends on b and c only. Stars which pass the meridian near the zenith are therefore the best to observe in order to determine the error of the clock.

70. Constants to be Determined. — Before the hour angle, t, can be computed by either of the formulæ (10), (11), or (12), the co-efficients a, b, c, d, and e, must be determined by observation. Equation (3) shows that t is expressed in time, hence the values found for a, b, c, d, and e, must also be expressed in time. Equations (8) and (9) show that d and e are known when a and b are given; hence it is only necessary to find by observation a, the *azimuth constant*, b, the *level constant*, and c, the *collimation constant*.

71. Azimuth Constant. — The value of a, the azimuth constant, may be found as follows:

Let the transits be observed of two stars differing considerably in declination, but which culminate nearly together. Let the observed times corrected for the errors of level and collimation, and if necessary for the rate of the clock, be denoted by T and T'. We then have $b=0$, $c=0$, and (12) becomes

$$t=a\,\frac{\sin(\phi-\delta)}{\cos\delta}.$$

which being substituted in (3) gives for one star,

$$\alpha = T + \Delta T + a\,\frac{\sin(\phi - \delta)}{\cos\delta};$$

and for the other,

$$\alpha' = T' + \Delta T + a\,\frac{\sin(\phi - \delta')}{\cos\delta'}.$$

Then, $$\alpha' - \alpha = T' - T + a\left\{\frac{\sin(\phi - \delta')}{\cos\delta'} - \frac{\sin(\phi - \delta)}{\cos\delta}\right\}$$

$$= T' - T + a\,\frac{\cos\phi\sin(\delta - \delta')}{\cos\delta\cos\delta'},$$

whence $$a = \frac{[(\alpha' - \alpha) - (T' - T)]\cos\delta\cos\delta'}{\cos\phi\sin(\delta - \delta')},$$

which may also be put in the form

$$a = \frac{(\alpha' - \alpha) - (T' - T)}{\cos\phi\,(\tan\delta - \tan\delta')} \tag{14}$$

If one of the observations be made on *a circumpolar star at its lower culmination,* its declination must be considered as measured over the pole; that is, it will be greater than 90°. We must then take for δ the supplement of the star's declination, and for α its right ascension increased by 12 hours. Substituting in (14) for α', $\alpha' + 12^h$, and for δ', $180° - \delta'$, it becomes

$$a = \frac{(\alpha' - \alpha) + 12^h - (T' - T)}{\cos\phi\,(\tan\delta + \tan\delta')} \tag{15}$$

If *the same star* be observed *at both upper and lower culmination,* we shall have $\alpha' = \alpha$, and $\delta' = \delta$, and (15) becomes

$$a = \frac{12^h - (T' - T)}{2\cos\phi\tan\delta} \tag{16}$$

The interval between the observations should be as small as possible, in order that the clock's rate may remain sen-

sibly constant. Hence the right ascensions of the two stars should differ as little as possible; or if one is observed at its lower culmination, they should differ by nearly 12 hours. Formula (16) will not give an accurate result unless the rate of the clock, as well as the instrumental adjustments, can be relied on for 12 hours.

We observe that the larger the factor $\tan\delta - \tan\delta'$ in the denominator of (14), the less will the errors of observation affect the value of a. Hence in the use of this formula, the best condition is that one star should be as near the pole, and the other as far from it, as possible. In the use of (15), however, the factor $\tan\delta + \tan\delta'$ will be the greater the nearer both stars are to the pole.

The application of either of the formulæ (14), (15), or (16) will give the value of a in seconds of *time;* multiplying this by 15, the result will be the azimuth deviation in seconds of arc.

As the errors of adjustment are regarded positive when they cause the middle wire to deviate from the meridian east of south, a negative value of a will indicate that the instrument deviates west of south, or east of north.

Example. *By Formula* (14).

$$
\begin{array}{lr@{\qquad}lr@{\qquad}l}
T' = & 19^h19^m46^s.62 & \alpha' = & 19^h19^m30^s.06 & \delta' = 2^\circ\,52'\,53'' \\
T = & 19\ 12\ 46\ .97 & \alpha = & 19\ 12\ 31\ .05 & \delta = 67\ \ 27\ \ 28 \\
\hline
T'-T = & 6\ 59\ .65 & \alpha'-\alpha = & 6\ 59\ .01 & \tan\delta' = 0.050 \\
\alpha'-\alpha = & 6\ 59\ .01 & & & \tan\delta = 2.409 \\
\hline
 & -\,0^s.64 & & & 2.359
\end{array}
$$

$$
\begin{array}{rl@{\qquad}rl}
\alpha'-\alpha-(T'-T) = & -\,0^s.64 & \log & \bar{1}.806180 \\
\phi = & 42^\circ\,43'\,52'' & \text{a. c. cos} & 0.133983 \\
\tan\delta - \tan\delta' = & 2.359 & \text{colog} & 9.627272 \\
a = & -\,0^s.37 & \log & \bar{1}.567435
\end{array}
$$

west of south.

72. Level Constant. — To find b, we observe that if the striding level is so adjusted that its axis is accurately parallel to the rotation axis of the telescope, both will be known to be horizontal when the two ends of the bubble read alike; but if the readings are different, their half-difference will denote the level error, b.

Let $e =$ reading of east end of level,

$w =$ reading of west end,

$s =$ value of one division of level scale in seconds of arc,

$b =$ inclination of rotation axis to horizon, positive if east end is too low.

Then
$$b = s\frac{w - e}{2} \tag{17}$$

As we cannot assume the adjustment of the level to be perfect, however, it should be reversed end for end, and the bubble read in both positions.

Let e' and e'' be the east readings, and w' and w'' the west readings; then will

$$e = \frac{e' + e''}{2} \text{ and } w = \frac{w' + w''}{2},$$

whence by (17),

$$b = s\frac{w' + w'' - (e' + e'')}{4};$$

or, expressed in time,

$$b = \frac{s}{60}[w' + w'' - (e' + e'')] \tag{18}$$

Example. — The level reads in one position, $e' = 14.9$, $w' = 21.5$, and after reversing, $e'' = 20.1$, $w'' = 16.3$; also $s = 4''.74$. Find the level constant in seconds of time.

Ans. $b = 0^s.22$.

73. Collimation Constant. — To find c, let the transit be observed of a close circumpolar star at its upper culmination. Note the times of its passage over the first half of the wires exclusive of the middle wire, then reverse the rotation axis, and note the times of passage over the same wires in reverse order. Find by means of the equatorial intervals, the time of transit over the mean wire from each set of observations, and denote the results by T and T'.

The value of t from (13) being substituted in (3), gives

$$\alpha = T + \Delta T + aA + bB + cC \tag{19}$$

After reversal of the axis the sign of c will be changed, and we shall have

$$\alpha = T' + \Delta T + aA + bB - cC.$$

The difference of these equations gives

$$T' - T = 2cC = \frac{2c}{\cos\delta},$$

or

$$c = \tfrac{1}{2}(T' - T)\cos\delta \tag{20}$$

74. Constants Determined by Equations of Condition. — The values of the constants a, b, and c, and that of the clock error ΔT, may also be determined by observing the transits of a number of stars whose places are well known, which differ little in right ascension but considerably in declination, and substituting the results for T in equation (19). From the known places we derive the values of α, A, B, and C for each star; hence, ΔT, a, b, and c are the only unknown quantities. Each observation thus furnishes an equation of condition, four of which will be sufficient; but for greater accuracy it will be better to increase the number of observations, and combine the resulting equations by the Method of Least Squares.

CHAPTER IV.

THE SEXTANT.

Description, Adjustment, and Use of the Sextant.

75. Description of the Sextant. — The sextant is an instrument designed for measuring the angular distance between two objects. The instrument, represented in Fig. 9, consists of the sector of a circle of about 15 inches diameter, with a graduated limb, and a movable radius carrying an index and vernier along the graduated arc, to any point of which it can be fixed by a clamp and tangent-screw. At the center of motion of this radius or arm, and

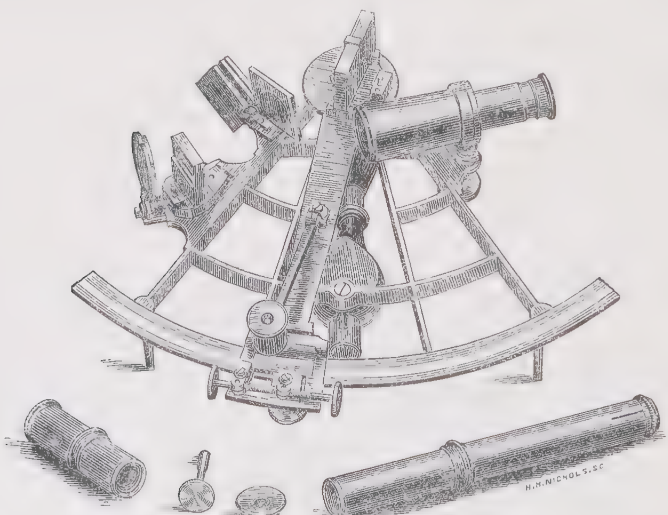

Fig. 9. — The Sextant.

moving with it, a small plane mirror called the *index-glass* is attached at right angles to the plane of the sector, and another mirror called the *horizon-glass* is permanently fixed to the instrument in such a position that the mirrors become parallel when the index stands at zero.

The horizon-glass is only one-half silvered, and a small telescope attached to the instrument has its axis directed nearly to the line which separates the silvered and unsilvered parts, and is thus able to receive at the same time reflected and direct rays. Thus, while the observer looks directly at one of the objects, by moving the index-arm with its mirror the image of the other object is brought into the field of view by reflection from both mirrors. The index now being clamped, the images of the two objects can be made to coincide by the tangent-screw, and the angle between them read off on the graduated arc.

As the two images should be of about equal brightness, the telescope is capable of being moved parallel to itself, nearer to or farther from the plane of the instrument, so as to receive more reflected or more direct rays.

The sector embraces about one-sixth of a circle, — whence the name sextant, — and the half-degree divisions on the arc are numbered as degrees, the reason for which is explained in the next article. Each so-called degree is subdivided into six equal parts, and the instrument reads by the vernier to $10''$.

In the focus of the telescope are two parallel wires equidistant from the optical axis. By turning the tube containing the eye-piece, the wires may be made parallel to the plane of the instrument. Their object is to mark the center of the field of view, and in all observations the images should be kept between them.

Behind each of the mirrors several dark glasses of different shades are hinged, any of which may be turned up to moderate the light when the sun is observed.

76. Double Reflection. — Let I and H, Fig. 10, be the two sextant-mirrors shown in section, S and S' the two objects whose angular distance is to be measured, and $SIHT$ the course of a ray of light from S, which, falling on the index-glass, is reflected to the horizon-glass, and again reflected into the telescope at T. Draw the lines CE and CH normal to the two mirrors; the angle C between the two normals is equal to the angle A between the two mirrors.

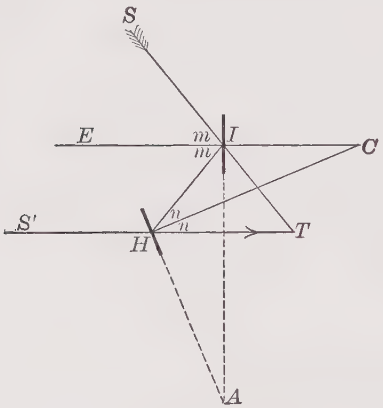

Fig. 10.

The angles of incidence and reflection at each point being equal, let those at the point I be denoted by m, and those at H by n. The angle EIH is exterior to the triangle CIH; hence we have

$$C = m - n.$$

The angle SIH is exterior to the triangle TIH; hence

$$T = 2m - 2n = 2C = 2A.$$

As the images of the two objects coincide, the second object, S', is seen directly in the line THS'; hence the angle between the objects is double the angle between the mirrors. For this reason half-degrees on the arc are called degrees, and an arc of 60° is sufficient to measure an angle of 120°.

77. Adjustment of the Sextant. — Both mirrors should be perpendicular to the plane of the sextant, and the axis of the telescope should be parallel to that plane. They are supposed to be properly adjusted by the maker, and are not

very liable to be thrown out of position, but the adjustment should occasionally be tested by the observer.

(1) The perpendicularity of the index-glass is tested by setting the index at about the middle of the arc, and noticing whether the reflected arc seen in the mirror appears to be a continuation of that seen directly. If the index-glass is not in adjustment, the arc will appear broken at the point where the direct and reflected parts come together.

(2) After proving the perpendicularity of the index-glass, that of the horizon-glass may be tested as follows:

Having set the index near zero, if the telescope be directed to a star or other distant object, two images will appear in the field, one direct, and the other reflected from the mirrors. Now, by moving the index-arm back and forth the reflected image should pass directly over the other. If it does not, but passes to one side of it, the horizon-glass is out of position, and may be adjusted by means of a screw at its back. The process should be repeated until the test is satisfied.

(3) To test the collimation axis of the telescope, place the sextant in the plane of two distant objects far apart, the sun and moon, for instance, and bring the reflected image of one in contact with the direct image of the other on one of the wires, then turn the instrument so as to bring the images on the other wire; if they are again in contact, the axis is in adjustment, for in these two positions it is equally inclined to the plane of the objects. If found out of adjustment, it may be brought back by two screws which hold in place the ring to which the telescope is attached.

When the telescope is in adjustment, if the images are made to coincide in the middle of the field, they will not coincide when brought on either of the wires. When the coincidence is made on either wire, the reading will be greater than when made midway between them; hence angles measured with the telescope out of adjustment are too great.

78. Method of Using the Sextant. — The sextant may be held in the hand, as it always is at sea, but on land it is better to mount it on a tripod to secure greater steadiness and consequent accuracy. The instrument must be held in the plane of the two objects whose angular distance is to be measured, the telescope being directed to the fainter one, in order that the brighter one may suffer the loss of light by double reflection. The two images should be made, as nearly as may be, equally bright. The images being brought nearly together, the index is clamped, and the contact made perfect by the tangent-screw. The reading of the vernier then gives the angular distance required.

Measurement of Altitudes with the Sextant.

79. Measuring the Sun's Altitude at Sea. — The sextant is principally used for measuring altitudes, particularly those of the sun. At sea the sun's altitude is measured by bringing the lower limb of its image reflected from the mirrors in contact with the visible horizon viewed directly through the telescope, the sextant being held vertical. Owing to the elevation of the observer above the sea-level, this observed altitude requires to be corrected for the dip of the horizon.

80. Artificial Horizon. — The natural horizon cannot be used for observations on land; hence an artificial horizon is used, consisting of a shallow vessel of mercury. To protect it from the wind, it should be covered with a roof made of two glass plates with surfaces exactly parallel, set at right angles to each other in a frame. With this arrangement we measure the angle between the sun or other body and its reflected image in the mercury. Since the surface of the mercury is horizontal, the image is as far below the horizon as the body is above; hence when the artificial horizon is used, the angle measured is the double altitude.

81. Altitude of the Sun's Center. — When the sun's altitude is measured, the observation is made on either the upper or the lower limb. The altitude of the center is then found by subtracting or adding the sun's *semi-diameter*, which is given for every day in the solar ephemeris.

In taking altitudes out of the meridian, it is better to take several near together and use their mean. In case of the sun, if we take an even number, alternately of the upper and lower limb, the mean of the whole is the double altitude of the center.

82. Observations on the Sun's Limbs. — The sun's image reflected in the mercury, viewed with the inverting telescope, is seen in its true position, being inverted twice, while the image reflected from the mirrors is inverted but once. Hence, to find the altitude of the sun's upper limb, we bring the lower limb of the latter image in contact with the upper limb of the former; and the reverse for the altitude of the sun's lower limb.

When the altitude is *increasing*, if the upper limb is observed, the images are approaching each other, and to find the altitude and the corresponding time we may set the images a little apart, with the index reading an even number of minutes, as 10′, 20′, 30′, etc., and note the time of exact contact. If the observation is on the lower limb, the images tend to separate, and we may overlap them a little and wait for contact. The process will be reversed when the altitude is *decreasing*.

83. Meridian Altitudes. — The meridian altitude is the highest altitude, and in measuring it, if the images are approaching, we merely keep them from overlapping by means of the tangent-screw, until they cease to approach each other, and then take the reading; and similarly if the images are separating.

Correction of Sextant Observations.

84. The Index Correction. — All angles measured with the sextant are to be corrected for *index error*.

When the two mirrors are parallel, the index should read zero; and if it does not, the reading is the index error.

The index *correction* is equal to the error, and of course has a contrary sign.

To find the index correction, set the index near 0, and direct the telescope to a distant object, as a star. Two images will be seen, one directly and the other by reflection from the mirrors. Make them coincide exactly by the tangent-screw, and read the index; the reading is the index correction.

The arc is graduated a few degrees to the right of 0. If the index stands on the right of 0, all the sextant readings are too small, hence the index correction is positive; but if it stands on the left of 0, the readings are too large, and the correction is negative.

85. The index correction can be more accurately found from an observation on the sun. Having set the index near 0, direct the telescope to the sun, and bring the direct and reflected images in contact, first with the index to the left of 0, and then to the right. The readings in opposite directions will have contrary signs; call that on the left, $-r$, and that on the right, $+r'$; and let $s =$ sun's apparent diameter, and $x =$ index correction.

When the centers of the images coincide, $r = x$; but when they are tangent externally,

$$r' = x + s,$$

and

$$-r = x - s,$$

half the sum of which gives

$$x = \tfrac{1}{2}(r' - r).$$

Hence the index correction is one-half the numerical difference of the readings, *plus* or *minus* according as the reading on the right of 0 is *greater* or *less* than that on the left.

If, for example, $r = 33' 55''$, $r' = 30' 40''$, we find

$$x = -\frac{3' 15''}{2} = -1' 37''.5.$$

86. The Correction for Refraction. — All altitudes measured with the sextant are to be corrected for refraction.

On account of the refraction of the atmosphere all bodies appear higher than they really are; hence the true altitude is less than the apparent altitude, which is that measured by the sextant. The amount of the refraction depends on the height of the body above the horizon, and, since changes in the pressure and temperature of the air affect its density, on the state of the barometer and thermometer.

The refraction tables give the amount of the *mean refraction* for any altitude; or in other words, that which corresponds to a barometric pressure of 30 inches, and temperature of 50° F. They also give the factors by which to multiply the *mean* in order to obtain the *true* refraction for the same altitude when the readings of the barometer and thermometer are different from these normal readings. See Table I.

The correction for refraction is always negative.

87. The Correction for Parallax. — Altitudes measured with the sextant, of all bodies except the fixed stars, are to be corrected for parallax.

The parallax of a body is the angle subtended by the radius of the earth passing through the observer, as seen from the body. As viewed from the earth's surface a body appears lower than it would if viewed from the center; hence the effect of parallax is the reverse of that of refraction, and the correction for parallax is always positive.

Like refraction, the amount of parallax depends on the altitude of the body, being zero at the zenith and a maxi-

mum — called horizontal parallax — at the horizon. The horizontal parallax of the sun, moon, or a planet, can be found for any day from the Nautical Almanac, and it remains to show how the parallax at any altitude may be found when the horizontal parallax is known.

88. Parallax in Altitude. — Let A, Fig. 11, be the place of the observer on the earth's surface, H the position of the body in the horizon, and B its position at any altitude h.

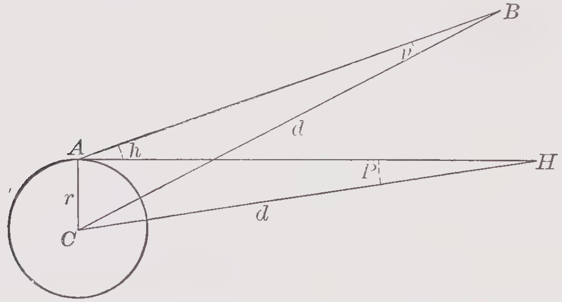

FIG. 11.

Let d denote the distance of the body from the center of the earth, r the earth's equatorial radius, P the horizontal parallax, and p the parallax in altitude.

In the right-angled triangle AHC we have

$$\sin P = \frac{r}{d};$$

and in the triangle ABC,

$$\frac{\sin p}{\sin(90° + h)} = \frac{r}{d} = \sin P,$$

whence $\quad \sin p = \sin P \sin(90° + h) = \sin P \cos h.$

For all bodies except the moon the parallax is very small, and we may put the angle in place of its sine, whence we have

$$p = P \cos h;$$

that is, the parallax in altitude is equal to the horizontal parallax multiplied by the cosine of the altitude.

CHAPTER V.

FINDING THE TIME BY OBSERVATION.

Time by Transit Observations.

89. Time of Meridian Passage. — By far the best method of finding the time is by observing the time of meridian passage of a celestial body with the transit instrument and clock.

Since the hour angle of a body on the meridian is zero, equations (9) and (7) of Chapter II. give

$$\textit{sidereal time of meridian passage} = \textit{body's R. A.} \tag{1}$$

$$\textit{mean time sun's meridian passage} = \textit{equation of time} \tag{2}$$

90. Civil and Astronomical Time. — The astronomical day begins at noon, and the civil day at the preceding midnight; hence the civil and astronomical dates differ by 12 hours. If the mean time clock is set to indicate *astronomical* time, the time of mean noon will be $00^h\,00^m\,00^s$; but if *civil* time, it will be $12^h\,00^m\,00^s$.

If civil time be used, equation (2) will be changed into

$$\textit{mean time sun's meridian passage} = 12^h + \textit{eq. of time} \tag{3}$$

91. The Clock Correction. — Finding the time by observation consists in finding the *error* or *correction* of the clock.

Let T = true time of meridian passage of a body,
T' = observed time of meridian passage,
x = clock correction;

then, $$x = T - T' \tag{4}$$

If x is *plus*, the clock is slow; if *minus*, the clock is fast.

The method of finding the value of T depends on the kind of time used.

(1) If sidereal time be used, we have from (1),

$$T = \textit{right ascension} \tag{5}$$

(2) If mean solar time be used, we have for a *star*, by equation (4), Chapter II.,

$$T = \textit{mean equivalent of } (R.\ A. - s') \tag{6}$$

and for the *sun*, by (2),

$$T = \textit{equation of time} \tag{7}$$

or, by (3),

$$T = 12^h + \textit{equation of time} \tag{8}$$

91 *a*. The Clock Rate. — The *rate* is the change of error of the clock. Suppose the error is found to be a, and after an interval of n days it is again found to be b, then the daily rate is $\frac{b-a}{n}$. If the clock is losing, the rate is positive; if gaining, it is negative.

Examples.

(1) *Observations in sidereal time.*

1. *α Lyræ.*

(1)	$18^h\ 32^m\ 13^s.4$
(2)	$34\ .5$
(3)	$33\ \ 01\ .3$
(4)	$28\ .0$
(5)	$49\ .1$
	$126\ .3$
	$.2$
	$25\ .26$
	$24\ .$

$$T' = 18^h\ 33^m\ 1^s.26$$
$$T\ = 18\ \ 33\ \ 7\ .63 = \text{R. A.}$$
$$x\ = \quad +6\ .37$$

The observations of the five wires are reduced thus: The sum of the seconds is taken, omitting the hours and minutes, and divided by 5, or multiplied by 0.2. As the minutes have been omitted, this is liable to differ from the true mean by one-fifth of a minute, that is, 12 seconds; or by any multiple of 12 seconds. Hence the mean of the seconds is to be corrected by such a multiple of 12 as will make it agree with the middle wire within a fraction of a second. The hours and minutes will be the same as for the middle wire.

	2. *α Aquilæ.*	3. *Pollux.*	4. *δ Draconis.*
R. A. =	$19^h\,45^m\,19^s.87$,	$7^h\,38^m\,20^s.54$,	$19^h\,12^m\,30^s.66$.
	$19^h\,45^m\,00^s.0$	$7^h\,37^m\,49^s.8$	$19^h\,10^m\,38^s.7$
	17 .0	38 8 .7	11 22 .0
	37 .9	32 .5	12 16 .2
	59 .0	56 .1	13 10 .5
	46 15 .9	39 15 .0	53 .8
$T' =$	19 45 37 .96	7 38 32 .42	19 12 16 .24
$T =$	19 45 19 .87	7 38 20 .54	19 12 30 .66
$x =$	− 18 .09	− 11 .88	+ 14 .42

5. *The Sun.* — *R. A.* = $6^h\,01^m\,51^s.98$.

	First limb.		*Second limb.*
	$5^h\,59^m\,51^s.8$		$6^h\,2^m\,9^s.9$
	6 00 9 .9		28 .0
	33 .0		50 .5
	55 .5		3 13 .5
	01 14 .0		31 .6
	5)164 .2		5)133 .5
	6 00 32 .84,	1st limb.	26 .70
	6 02 50 .70,	2d limb.	24 .
$T' =$	6 01 41 .77	= mean.	6 2 50 .70
$T =$	6 01 51 .98	= R. A.	
$x =$	+ 10 .21		

The same result may be found in another way as follows: If from the sun's right ascension we subtract for the first limb, and add for the second the sidereal time required for the semi-diameter to pass the meridian, we shall have the computed times of transit of the two limbs. Thus:

	I.	II.
Sun's R. A. =	$6^h\,01^m\,51^s.98$	$6^h\,01^m\,51^s.98$
Passage semi-diam. =	1 8 .97	1 8 .97
$T =$	6 00 43 .01	6 03 00 .95
$T' =$	6 00 32 .84	6 02 50 .70
$x =$	+ 10 .17	+ 10 .25

The mean of these results is $10^s.21$ as before.

(2) *Observations in mean solar time.*

6. ζ *Virginis*, June 15.

	h	m	s
	7	52	34 .9
			51 .35
			59 .7
		53	8 .0
			24 .65
$T' =$	7	52	59 .72
$T =$	7	52	58 .04
$x =$			− 1 .68

7. η *Ursa Major*, June 19.

	h	m	s
	7	50	52 .8
		51	12 .3
			24 .9
			37 .9
			57 .3
$T' =$	7	51	25 .04
$T =$	7	51	23 .34
$x =$			− 1 .70

8. γ *Aquilæ*, Nov. 9.

	h	m	s
	4	25	8 .5
			12 .55
			17 .0
			25 .25
			29 .4
			33 .8
			37 .9
			42 .25
			50 .4
			54 .85
			59 .2
$T' =$	4	25	33 .73
$T =$	4	26	0 .32
$x =$			+ 26 .59

9. θ *Aquilæ*, Nov. 9.

	h	m	s
	4	49	40 .2
			44 .25
			48 .4
			56 .9
		50	1 .1
			5 .15
			9 .25
			13 .3
			21 .85
			25 .75
			30 .1
			29 .11
			24 .
$T' =$	4	50	5 .11
$T =$	4	50	31 .71
$x =$			+ 26 .60

The values of T are computed in examples 9, 10, 11, and 12, Art. 43.

The reduction of observations over eleven wires is simplified by omitting the middle wire, taking the mean of the seconds of the remaining ten, and correcting it by any multiple of 6 if necessary, in order to make it agree with the seconds of the middle wire.

10. *The Sun.—Equation of Time* $= -15^m\,56^s.82$.

I.	*Civil Times.*	II.
$11^h\,42^m\,28^s.5$		$11^h\,44^m\,40^s.6$
45 .2		57 .7
43 6 .8		45 19 .0
28 .2		40 .1
45 .1		57 .1
11 43 6 .76, 1st limb.		11 45 18 .90
11 45 18 .90, 2d limb.		

$$T' = 11\ \ 44\ \ 12.83 = \text{mean.}$$
$$T\ = 11\ \ 44\ \ \ 3.18$$
$$x\ = \qquad -9.65$$

By formula (8):

$$12^h\,00^m\,00^s.00$$
$$\text{Eq. time} = \ -15\ \ 56.82$$
$$T = 11\ \ 44\ \ \ 3.18$$

11. *The Sun.—Equation of Time* $= -3^m\,10^s.28$.

I.	*Civil Times.*	II.
$11^h\,55^m\,16^s.0$		$11^h\,57^m\,27^s.9$
33 .0		44 .9
54 .6		58 6 .5
56 16 .0		28 .0
33 .3		45 .1

We find

$$T' = 11^h\,57^m\,00^s.53$$
$$T\ = 11\ \ 56\ \ 49.72$$
$$x\ = \qquad -10.81$$

12. *The Sun.* — *Equation of Time* = $2^m\,3^s.53$

I.	*Astronomical Times.*	II.
$0^h\,0^m\,42^s.8$		$0^h\,2^m\,52^s.1$
47 .25		56 .3
51 .4		3 0 .6
59 .2		9 .0
1 4 .25		13 .2
8 .85		17 .5
12 .35		21 .6
16 .85		25 .9
25 .6		34 .5
29 .6		38 .6
33 .7		43 .1

We find

$$\begin{aligned} T' &= 0^h\,2^m\,12^s.89 \\ T &= 0\;\;2\;\;\;\,3\,.53 \\ x &= \;\;\;\;\;\; -9\,.36 \end{aligned}$$

92. Standard Time. — Since November, 1883, *standard* mean time has been in general use in the United States instead of *local* mean time. The time of the meridian 5 hours or 75° west of Greenwich is called Eastern time, and is the standard time for all places whose west longitudes are between $4^h\,30^m$ and $5^h\,30^m$; Central time, viz. that of the meridian 6 hours or 90° west is the standard time for places between $5^h\,30^m$ and $6^h\,30^m$; Mountain time, that of the meridian 7 hours or 105° west is the standard time between $6^h\,30^m$ and $7^h\,30^m$; Pacific time, that of the meridian 8 hours or 120° west is the standard time between $7^h\,30^m$ and $8^h\,30^m$. Thus the standard and local mean time at any place can never differ by more than half an hour.

If the clock whose correction is required is regulated to standard time, the mean time computed from formulæ (6), (7), and (8) must also be reduced to standard time.

The following is an observation in Eastern time, which is $5^m\,17^s.66$ slower than local mean time:—

13. *The Sun.* — *Equation of time* $= -14^m\,42^s.56$.

I.	II.
$11^h\,38^m\,44^s.8$	$11^h\,40^m\,55^s.3$
48.7	59.3
53.15	41 3.7
39 1.2	12.1
5.75	16.3
9.7	20.4
13.95	25.0
18.4	28.9
27.15	27.3
31.25	41.6
35.25	46.0

	11 39 9.96, 1st limb.			
	11 41 20.55, 2d limb.			
$T' =$	11 40 15.51 = mean.			
$T =$	11 39 59.78			
$x =$	−15.73			

By formula (8):

		$12^h\,00^m\,00^s.00$
Eq. of time	=	− 14 42.56
Civil time	=	11 45 17.44
		5 17.66
Standard time	=	11 39 59.78

Time by Equal Altitudes.

93. Equal Altitudes of a Star. — If equal altitudes of a star east and west of the meridian, and the corresponding times, be observed by the sextant and sidereal clock, and if t and t' denote the observed times, we shall have by equation (12), Chapter II.,

$$T' = \tfrac{1}{2}(t + t').$$

We also have by (5),

$$T = \text{star's R. A.},$$

and by (4),

$$x = T - T'.$$

As the altitudes themselves are not used, the corrections for index error and refraction are not required.

94. Equal Altitudes of the Sun. — If the times corresponding to equal altitudes of the sun be observed, their half sum will not be the time of meridian passage, but will require a correction on account of the sun's change of declination during the interval.

To find what effect the change of declination has on the sun's hour angle, take the general equation (8), Chapter I.,

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos P,$$

and differentiate it on the supposition that δ and P are the only variables; we find

$$0 = \sin \phi \cos \delta \, d\delta - \cos \delta \cos \phi \sin P dP - \cos \phi \cos P \sin \delta \, d\delta,$$

whence

$$d\delta \, (\sin \phi \cos \delta - \cos \phi \sin \delta \cos P) = \cos \phi \cos \delta \sin P dP,$$

and

$$dP = d\delta \left(\frac{\sin \phi}{\cos \phi \sin P} - \frac{\sin \delta \cos P}{\cos \delta \sin P} \right),$$

which reduces to

$$\frac{dP}{d\delta} = \left(\frac{\tan \phi}{\sin P} - \frac{\tan \delta}{\tan P} \right) \tag{9}$$

95. Equation of Equal Altitudes. — Let

t = half the interval between the observations, expressed in hours;

Δ = sun's hourly change of declination;

e = correction to be added to the mean of the observed times.

The value of Δ may be taken from the solar ephemeris, and will be *plus* if the declination is increasing, and *minus* if decreasing. If we suppose it to be *positive*, the hour angle west of the meridian will be greater than that east of the meridian, and the required correction, e, will be *negative*. For a similar reason, if Δ is *negative*, e will be *positive*.

The product $t\Delta$ is the sun's change of declination during half the interval, and e is the change it produces in the hour angle. These quantities are very small, and have sensibly the same relation as $d\delta$ and dP, hence we may substitute $-\frac{e}{t\Delta}$ for $\frac{dP}{d\delta}$.

By equation (11), Chapter II., we also have, with sufficient accuracy, $P = t$.

Making these substitutions in equation (9), and dividing by 15 to get the result in *time*, we have

$$e = -\frac{t\Delta}{15}\left(\frac{\tan\phi}{\sin t} - \frac{\tan\delta}{\tan t}\right) \qquad (10)$$

If we put $\quad \frac{-t}{15\sin t} = A,$ and $\frac{t}{15\tan t} = B,$

(10) becomes $\quad e = A\Delta\tan\phi + B\Delta\tan\delta \qquad (11)$

the equation of equal altitudes.

The logarithms of A and B are computed for different values of the interval $2t$, and may be taken from Table II.

The half sum of the observed times corrected by the equation of equal altitudes gives the observed time of the sun's meridian passage. The *true* time is given by equation (7) or (8), and the difference is the clock correction.

EXAMPLE.

A. M. times.	*Double alt.*	*P. M. times.*
$9^h\,21^m\,07^s$	$64°\,00'$	$2^h\,15^m\,04^s$
21 48	10	14 25
22 27	20	13 46
25 45	65 10	10 29
26 24	20	9 48
29 07	66 00	7 07
30 25	20	5 47
31 09	30	5 05
31 49	40	4 26
32 29	50	3 43
10)272 30		10)89 40
9 27 15		14 8 58
		9 27 15
		$2t =$ 4 41 43

$12^h\,00^m\,00^s.00$	2)23 36 13
Eq. time $= -11\ 51\ .35$	$\frac{1}{2}$ sum $= 11\ 48\ \ 6.5$
$T = 11\ 48\ \ 8\ .65$	$e = \ \ +\ 15.61$
$T' = 11\ 48\ 22\ .11$	$T' = 11\ 48\ 22.11$
Clock 13 .46 fast.	

Computation of $e = A\Delta \tan\phi + B\Delta\tan\delta$.

$\log A,$	-9.4337	$\log B,$	9.3457
$\Delta = -57''.65$ log	-1.7608	$\log\Delta,$	-1.7608
$\log\tan\phi,$	9.9656	$\delta = -5°\ 9'\ 50''$ tan	-8.9560
$e' = 14^s.46$ log	$+1.1601$	$e'' = 1^s.15$ log	$+0.0625$
$e'' = \ \ 1\ .15$			
$e\ = 15^s.61$			

TIME BY A SINGLE ALTITUDE.

96. Formula for the Hour Angle. — Equation (26), Chapter I. may be written in the form

$$\sin\tfrac{1}{2}P = \sqrt{\frac{\sin\frac{1}{2}[z-(\phi-\delta)]\sin\frac{1}{2}[z+(\phi-\delta)]}{\cos\delta\cos\phi}} \qquad (12)$$

In this equation, the latitude of the place, (ϕ), is supposed to be known, the zenith distance of the body, (z), is the

complement of the measured altitude, and the declination, (δ), is to be taken from the Nautical Almanac, observing that north declinations are to be marked $+$, and south $-$.

The altitude may be measured with the sextant, and the time of the observation must be noted by the clock.

We find from equation (12) the value of the hour angle P, from which the time of the observation is computed by means of equation (7) or (9), Chapter II. The difference between this and the *observed* time is the clock error.

Example.

Double altitudes of the sun's upper limb and corresponding observed mean times. Barometer, 30.0 in.; thermometer, 40°.

	Observed times.	*Double Altitudes.*	
	$21^h\ 48^m\ 01^s$	58° 30′	
	48 54	58 40	
	51 31	59 10	
	52 22	59 20	
	53 18	59 30	
	54 09	59 40	
	55 54	60 00	
	56 50	60 10	
	57 45	60 20	
	58 38	60 30	
	10)532 322	10)592 230	
	$21^h\ 53^m\ 44^s.20$	59° 35′ 00″.0	
E. long. =	13 29.75	− 9 52 .5	= index cor.
	21 40 14.45	2)59 25 7 .5	[U. L.
$= 24^h - 2^h.33 =$ Wash. time.		29 42 33 .75	= app. alt. of
		1 44 .05	= refraction
		29 40 49 .70	
		7 .74	= parallax
		29 40 57 .44	
		16 8 .16	= semi-di.
		29 24 49 .28	= alt. center
		90	
		60 35 10 .72	$= z$.

Computation by Equation (12) : —

$\phi =$	$42° 43' 53''$	cos 9.866017
$\delta =$	$-12\ 18\ 45$	cos 9.989894
$\phi - \delta =$	55 02 38	19.855911
$z =$	60 35 11	
$z - (\phi - \delta) =$	5 32 33	
$\frac{1}{2}$ do. $=$	2 46 17	sin 8.684406
$z + (\phi - \delta) =$	115 37 49	
$\frac{1}{2}$ do. $=$	57 48 55	sin 9.927542
		18.611948
		19.855911
		2)18.756037
$\frac{1}{2} P =$	$-13° 48' 54''.8$	sin 9.378019

		24^h
$P = -27° 37' 49''.6$	$= -$	$1^h\ 50^m\ 31^s.3$
Apparent time	$=$	22 09 28 .70
Equation of time	$=$	-15 53 .73
Mean time	$=$	21 53 34 .97
Observed time	$=$	21 53 44 .20
Clock fast,		9 .23

97. Best Position for Time Observations. — Equations (5) and (8), Chapter I., are

$$\cos h \sin Z = \cos \delta \sin P \tag{13}$$

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos P \tag{14}$$

The differentiation of (14) with respect to h and P gives

$$\cos h\, dh = -\cos \delta \cos \phi \sin P dP,$$

whence $$dP = -\frac{\cos h\, dh}{\cos \delta \cos \phi \sin P};$$

which, by (13), reduces to

$$dP = -\frac{dh}{\cos \phi \sin Z} \tag{15}$$

As ϕ is constant, dP is a minimum when $\sin Z$ is a maximum $= 1$, or $Z = 90°$; that is, when the body is on the prime vertical. In this position a small error in the measured value of h will have the least effect on the hour angle P. Altitudes measured for the purpose of finding the time should therefore be taken when the body is in or near the prime vertical.

98. Times of Rising and Setting. — Equation (16), Chapter I., for the hour angle of a body in the horizon, is

$$\cos P = -\tan \phi \tan \delta,$$

and shows that when the declination is *south*, the hour angle is in the first quadrant, that is, *less than six hours;* and when *north*, the hour angle is in the second quadrant, that is, *greater than six hours.*

Knowing the hour angle, the time of true rising or setting will be found by equation (7) or (9), Chapter II.

To find the time of *apparent* rising or setting, that is, the time when a body appears in the horizon, we use equation (12), making

$$z = 90° + \text{refraction} = 90° \, 34' \, 30'',$$

since the body at this time is really $34\frac{1}{2}'$ below the horizon.

If the time of apparent rising or setting of the *sun's upper limb* be required, it is necessary to make in equation (12),

$$z = 90° + \text{refraction} + \text{semi-diameter} = 90° \, 50' \text{ nearly.}$$

CHAPTER VI.

FINDING DIFFERENCES OF LONGITUDE.

99. The *difference of longitude* of two places on the earth's surface is the arc of the celestial equator included between their meridians. If the meridian of one of the places be assumed as the *first meridian,* their difference of longitude is called *the longitude* of the other place.

Longitude by the Telegraph.

100. **Exchange of Time Signals.** — All astronomical methods of finding differences of longitude depend on the principle that at any given instant, *the difference of local time at any two places is the same as their difference of longitude expressed in time.*

The comparison of local times can be best accomplished by the transmission of telegraphic signals between the two stations. Suppose a signal to be sent from the eastern station, E, to the western, W.

Let $e =$ time at E of sending the signal,

$w =$ time at W of receiving the signal,

$\lambda =$ difference of longitude of E and W.

Then if we neglect the small interval of time between the instant at which the signal is sent and that at which it is received, we shall have

$$\lambda = e - w \tag{1}$$

Let us denote by τ, the small interval occupied in the transmission of the signal, then the time, w, of receiving

the signal is too late by τ, hence w must be replaced by $w - \tau$ in equation (1), giving

$$\lambda = e - (w - \tau) = e - w + \tau \qquad (2)$$

The unknown term τ may be eliminated by sending another signal in the opposite direction.

Let $\qquad w' =$ time at W of sending signal,

$\qquad\qquad e' =$ time at E of receiving signal;

then, neglecting time of transmission,

$$\lambda = e' - w' \qquad (3)$$

But the time, e', of receiving the signal is too late by τ, and e' must be replaced by $e' - \tau$, giving

$$\lambda = (e' - \tau) - w' = e' - w' - \tau \qquad (4)$$

The half sum of (2) and (4) is

$$\lambda = \tfrac{1}{2}(e - w + e' - w') \qquad (5)$$

We have supposed a single telegraphic signal to be sent in each direction, but in practice, in order to secure the highest accuracy, a series of signals is sent in both directions each evening, and the operation is repeated on a number of nights in succession.

101. Observations for Clock Corrections. — As it is assumed that the local times of giving and receiving the signals are accurately known, the exchange of signals should be preceded and followed by a series of observations by each observer, for the determination of his clock correction. These observations should be so arranged as to make known the required corrections with the highest attainable accuracy.

In the longitude work of the United States Coast and Geodetic Survey the following arrangement is followed on those nights on which time signals are exchanged: A

set of ten star transits is observed, five of them with the axis of the instrument in one position, and the other five with the axis reversed. Of these five stars, four are circumzenith stars, two of which culminate north and two south of the zenith, and one is a circumpolar star. Two such sets of ten stars are observed on each night, and the exchange of signals takes place between them.

102. Method of Sending Signals. — The signals sent may be either arbitrary or automatic. In the first method, the observer at E taps his signal-key a number of times at short intervals, the time of each signal being recorded on the chronographs at both stations; then the observer at W gives a series of taps, the times being recorded on both chronographs. The chronograph records thus furnish the means of comparing the times of giving and receiving both sets of signals.

In the other method, the observer at E places his clock in the circuit and allows it to record its beats on the chronographs at both stations; then the observer at W places his clock in the circuit, and its beats are recorded on both chronographs. The mean of the records of both chronographs, corrected for clock errors, then gives the difference of longitude of the stations.

103. Effect of Personal Equation. — If the clock errors are well determined and the exchange of signals is conducted with sufficient care, the greatest error affecting the result will be that due to personal equation. This may be in great part eliminated by the observers' changing their places after half the series of observations is completed. The difference of the mean results obtained before and after changing places will be double the remaining effect of personal equation, and the corrections may be applied, with contrary signs, to the results obtained before and after making the change.

Example.

The following results of a series of exchanges of signals between San Francisco and a station near the Lick Observatory, Mt. Hamilton, Cal., by observers of the United States Coast and Geodetic Survey in the autumn of 1888, are given by permission of the Superintendent of the Survey as an illustration of this method.

Date, 1888.	Eastward Signals.	Westward Signals.	Mean of East and West Sig.	Difference of Longitude.
Oct. 23	$3^m09^s.099$	$3^m09^s.076$	$3^m09^s.088$	$3^m08^s.944$
30	.180	.148	.164	09.020
31	.138	.128	.133	08.989
Nov. 1	.263	.259	.261	09.117
2	.221	.213	.217	09.073
5	.248	.244	.246	09.102
Observers change places.			Mean = 3 09 .185	
Nov. 23	3 08 .899	3 08 .894	3 08 .896	09.040
24	.885	.864	.874	09.018
26	.953	.935	.944	09.088
27	.910	.902	.906	09.050
28	.875	.857	.866	09.010
			Mean = 3 08 .898	3 09 .041
			3 09 .185	
			2)0.287	
			Personal Eq. Cor. = 0.144	

The numbers in the second column are the values of $e' - w' - \tau$ from the San Francisco signals sent eastward, and those in the third column the values of $e - w + \tau$ from the Mt. Hamilton signals. Those in the fourth column are the values of $\frac{1}{2}(e - w + e' - w')$ for each night of observa-

tion. The values obtained before the observers change places are found to be the largest, and the personal equation correction is ∓ 0.144, which being applied with the upper sign to the results found before changing, and the lower sign to those after, give the corrected results in the last column, the mean of which gives finally,

$$\lambda = 3^{m}\,09^{s}.041.$$

Longitude by Transportation of Chronometers.

104. The difference of local times of two places, A and B, and therefore their difference of longitude, may be found by adjusting a chronometer to the local time at A, then carrying it to B and comparing it with the local time at that place. It is of course necessary to know and take into account the rate of the chronometer during the interval of its passage.

By a series of transit observations at intervals of several days, let the daily rate, ρ, be determined, then find the error, ϵ, of the chronometer on local time at A. Now transport the chronometer to B, and determine the error, ϵ', on local time at B. Let t be the time in days between the two determinations, ϵ and ϵ', then, when the error on B time is ϵ', that on A time will be $\epsilon + t\rho$, and the difference of these errors is of course the difference of local times at A and B. Hence we have

$$\lambda = \epsilon + t\rho - \epsilon' \tag{6}$$

This assumes that the rate is constant, and the same while being carried as when at rest. In fact, this is not true, and it is customary in practice to transport a number of chronometers as checks upon each other, using the mean of the results given by each separately.

This method of finding differences of longitude is not much used except at sea.

Longitude by Moon Culminations.

105. This method depends on the moon's rapid motion in right ascension, which, on an average, is a little over two minutes an hour. Hence if the moon and a certain star should pass the meridian of the given place at the same moment, the times of their passing the meridian one hour or 15° west of this would differ about two minutes, the times of their passing the meridian two hours or 30° west would differ about four minutes, and so on. If, then, at any two places the interval between the culminations of the moon and a given star be observed, the difference of those intervals, divided by the moon's hourly change of right ascension, will give the difference of longitude of the two places, expressed in hours.

If observations are made at only one place, the longitude of that place may still be found by comparing the moon's right ascension at the time of its observed culmination, or what is the same thing, the sidereal time at that instant, with the sidereal time at the first meridian, found from the tables of the moon in the Nautical Almanac.

106. Observations for Finding the Moon's Right Ascension. — To find the moon's right ascension at the time of transit, the observer notes the sidereal times of meridian passage of the moon's bright limb, and that of a star whose right ascension is known; the interval will be their difference of right ascension, which being added to or subtracted from that of the star will give that of the moon's limb. The right ascension of the center is then found by adding to that of the first limb, or subtracting from that of the second, the sidereal time required for the semi-diameter to pass the meridian.

Now, let

$\alpha =$ R. A. of moon's center at time of transit,

that is,

$$\alpha = \text{local sidereal time at instant of transit;}$$

and let

$$S = \text{Greenwich sidereal time at same instant;}$$

then

$$\lambda = S - \alpha \qquad (7)$$

The accuracy of the determination of α will be increased by observing several stars in connection with the moon, a part of which culminate before, and a part after it. They should be situated near the moon, because the errors of the transit instrument and clock will then be practically the same for all the observations, and will be eliminated in finding the observed intervals.

107. Use of the Moon's Hourly Ephemeris. — It remains to show how to find the value of S in equation (7).

The Nautical Almanac gives the moon's right ascension and declination for every hour of Greenwich mean time throughout the year. Let us then put

$\alpha_1 =$ R. A. next less than α, given in the ephemeris,
$M =$ Greenwich mean time corresponding to α,
$M_1 =$ Greenwich mean time corresponding to α_1,

then

$\alpha - \alpha_1 =$ change of R. A. in the interval $M - M_1$;

also let

$\Delta =$ increase of moon's R. A. in one minute at the time M_1, expressed in seconds,

and $d =$ hourly increase of Δ.

Now the second difference (d) will be found to be sensibly constant, hence the first difference (Δ) varies uniformly, and *its mean value for any interval is the value which it has at the middle of that interval.* Let the interval $(M - M_1)$ be expressed in seconds, then the mean value of Δ for this interval will be

$$\Delta + \tfrac{1}{2}(M - M_1)\frac{d}{3600}.$$

This is the average increase of right ascension in one minute during the interval, and being multiplied by the number of minutes in the interval gives the total increase, $\alpha - \alpha_1$. That is, we have

$$\left\{ \Delta + \tfrac{1}{2}(M - M_1)\frac{d}{3600} \right\} \frac{M - M_1}{60} = \alpha - \alpha_1;$$

or putting $M - M_1 = x$,

$$\left(\Delta + \frac{xd}{7200} \right) \frac{x}{60} = \alpha - \alpha_1;$$

whence

$$x = \frac{60(\alpha - \alpha_1)}{\Delta \left(1 + \frac{x}{7200} \cdot \frac{d}{\Delta} \right)} = \frac{60(\alpha - \alpha_1)}{\Delta} \left(1 + \frac{x}{7200} \cdot \frac{d}{\Delta} \right)^{-1}$$

$$= \frac{60(\alpha - \alpha_1)}{\Delta} \left(1 - \frac{x}{7200} \cdot \frac{d}{\Delta} \right), \text{ approximately.}$$

Let $$\frac{60(\alpha - \alpha_1)}{\Delta} = x' \qquad (8)$$

then

$$x = x' - \frac{xx'}{7200} \cdot \frac{d}{\Delta} = x' - \frac{x'^2}{7200} \cdot \frac{d}{\Delta}, \text{ nearly,}$$

since x' is an approximate value of x. If we let

$$\frac{x'^2}{7200} \cdot \frac{d}{\Delta} = x'' \qquad (9)$$

then $$x = x' - x'',$$

that is, $$M = M_1 + x' - x'' \qquad (10)$$

This value of M is the Greenwich mean time at the instant of observed moon transit, and S, the Greenwich sidereal time at the same instant, will be found by equation (6), Chapter II.

Example.

Observed sid. time transit $\begin{cases} \text{Moon's 2d limb} = 6^h\,22^m\,53^s.78 \\ \mu\ \textit{Geminorum} \quad = 6\ \ 14\ \ 56\ .40 \end{cases}$

Difference R. A.	=	7 57 .38
Star's R. A.	=	6 16 5 .00
R. A. moon's 2d limb	=	6 24 2 .38
Semi-diameter interval	=	1 13 .42
R. A. moon's center $= \alpha$	=	6 22 48 .96
$M_1 = 17^h$. . . . . α_1	=	6 22 12 .25
$\Delta = 2^s.6486$,	$\alpha - \alpha_1$ =	36 .71
$d = 0^s.0005$.		

60	log	1.778151
$\alpha - \alpha_1 = 36^s.71$	log	1.564784
$\Delta = 2^s.6486$	colog	9.576984
$x' = 831^s.61$	log	2.919919
		2
$\log x'^2 = 2 \log x' =$		5.83984
$d = 0.0005$	log	$\bar{4}.69897$
7200	colog	6.14267
$\Delta = 2.6486$	colog	9.57698
$x'' = 0^s.018$	log	$\bar{2}.25846$

By equation (10),	$M =$	$17^h\,13^m\,51^s.59$
Sid. time Greenwich mean noon	=	18 00 52 .22
(Table *A*.)	$cM =$	2 49 .84
	$S =$	11 17 33 .65
	$\alpha =$	6 22 48 .96
	$\lambda =$	4 54 44 .69

CHAPTER VII.

FINDING THE LATITUDE OF THE PLACE.

108. The latitude of a place, (ϕ), considered astronomically, is the arc of the meridian ZC, Fig. 3, included between the zenith of the place and the equator. The co-latitude, (ψ), is the arc ZP between the zenith and the pole. Now, since ZA, ZB, and PC are quadrants, we have $AP = CZ$, and $BC = PZ$; that is, the altitude of the pole above the horizon is equal to the latitude, and the altitude of the equator is equal to the colatitude.

The following are some of the principal methods of determining the latitude of a place.

Latitude by a Circumpolar Star.

109. Let S_2 and S_3, Fig. 3, be the points of culmination, and let $h = AS_2$, $h' = AS_3$, the greatest and least altitudes. We have

$$p = PS_2 = PS_3 = \text{star's polar distance};$$

also, $$AP = AS_2 - PS_2 = AS_3 + PS_3,$$

or $$\phi = h - p \text{ and } \phi = h' + p;$$

one-half the sum of which is

$$\phi = \tfrac{1}{2}(h + h') \qquad (1)$$

that is, the latitude is one-half the sum of the greatest and least altitudes.

The measured altitudes must be corrected for refraction.

Latitude by a Meridian Altitude or Zenith Distance.

110. **Formulæ for the Latitude.** — From Art. 33 we may derive the formulæ for the latitude in terms of the meridian altitude or zenith distance, and the declination or polar

distance; the former being found by observation, and the latter from the tables.

(1) If the body culminates *south of the zenith,* we have, from (18) and (21), Chapter I.,

$$\phi = \delta + z \tag{2}$$

$$\psi = h - \delta \tag{3}$$

(2) If *between the zenith and pole,* from (19) and (22),

$$\phi = \delta - z \tag{4}$$

$$\phi = h - p \tag{5}$$

(3) If *below the pole,* from (20) and (23),

$$\psi = z - p \tag{6}$$

$$\phi = h + p \tag{7}$$

The measured altitude, or zenith distance, is always to be corrected for refraction, and, unless the body observed be a fixed star, for parallax and semi-diameter.

Example.

Meridian altitude of the sun's upper limb measured with the sextant. Barometer reading, 29.9 inches; thermometer, 55°.

Observed double altitude	=	77° 01′ 10″
Index correction	=	− 9 45
		2)76 51 25
Apparent altitude upper limb	=	38 25 42 .50
Refraction	=	1 12 .60
		38 24 29 .90
Parallax	=	6 .96
		38 24 36 .86
Semi-diameter	=	16 5 .77
Altitude center	=	38 8 31 .09
		90
	$z =$	51 51 28 .91
	$\delta =$	− 9 7 37 .27
By equation (2),	$\phi =$	42° 43′ 51″.64

Latitude by the Zenith Instrument. — Talcott's Method.

111. This method of finding the latitude is the invention of Capt. Andrew Talcott, late of the United States Corps of Engineers. Its superiority over all other methods is due to the fact that it substitutes the measurement of a small distance in the field of view by the micrometer for that of a large arc by means of a graduated circle.

112. Description and Use of the Zenith Instrument. — Suppose two stars to culminate, one south and the other north of the zenith. By equations (2) and (4) we have for the two stars, respectively,

$$\phi = \delta + z, \qquad \phi = \delta' - z';$$

and taking their half sum,

$$\phi = \tfrac{1}{2}(\delta + \delta') + \tfrac{1}{2}(z - z') \tag{8}$$

The first term may be found from the star tables, but the last term must be measured. The *zenith instrument*, represented in Fig. 12, is designed especially for this purpose. Like the transit instrument, it is used only in the meridian. The telescope is capable of motion about either a horizontal or a vertical axis, and the observation is made on the two stars in succession by revolving it 180° about the vertical axis, its inclination remaining unchanged. In order that the two stars may appear successively in the field of view, they must be situated at nearly equal distances from the zenith. To avoid making observations near the edge of the field the difference of zenith distance should not exceed 15′ or 20′.

The telescope having been adjusted in the meridian, two stops are clamped to the horizontal circle in such position that the instrument may be turned 180° about the vertical axis, but will be brought to a stand in the plane of the

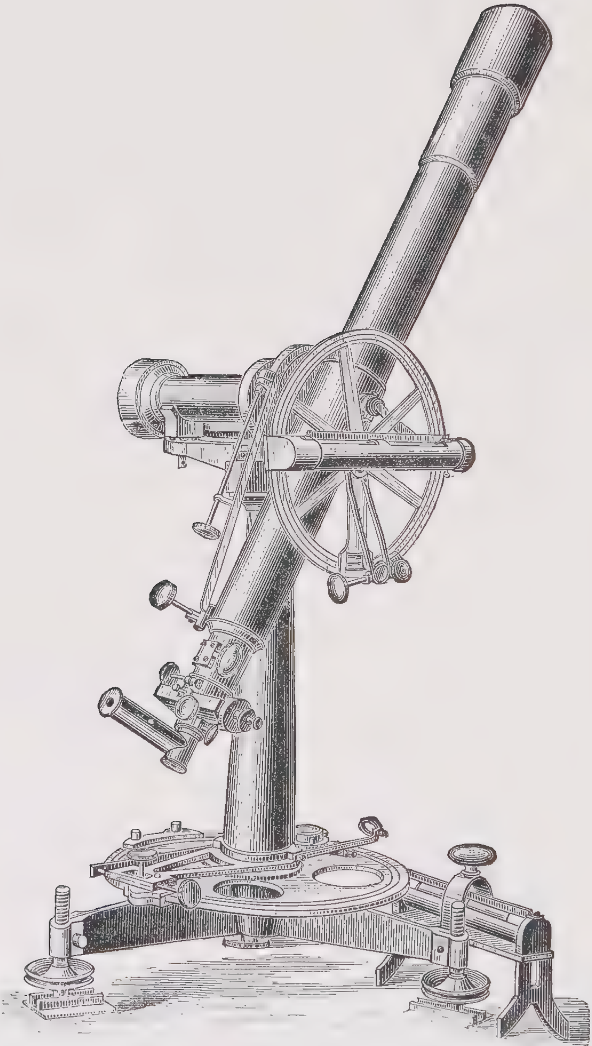

FIG. 12.—THE ZENITH INSTRUMENT.
(By Fauth & Co., Washington, D.C.)

meridian when the clamp of the horizontal circle bears against either stop.

The level attached to the finding circle of the telescope is much more delicate than those of the transit instrument, in order to measure the slightest change in the position of the vertical axis. This is an essential feature of the instrument, and one on which the accuracy of this method largely depends.

The measurement of the difference of zenith distance $(z-z')$, is made by a micrometer; a movable horizontal wire, represented by the dotted line in Fig. 13, being carried up and down parallel to itself by means of a fine motion screw. The number of entire revolutions of the screw is read from a scale in the field of view, the smallest divisions of the scale denoting single revolutions. The value of one revolution in seconds of arc being determined, a motion of the wire caused by any number of revolutions of the screw will be known in seconds. The head of the screw is divided into 100 equal parts, and one-tenth of each part may be easily estimated, so that a motion of the wire corresponding to one-thousandth of a revolution can be measured. In this way readings may be taken as small as 0.″05.

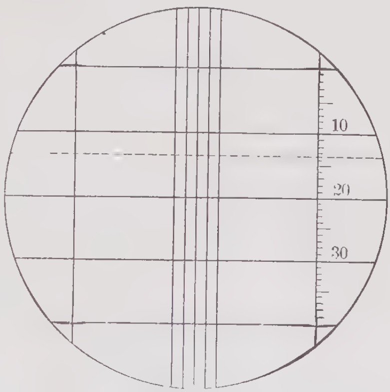

Fig. 13.

Five vertical wires are placed in the middle of the field of view, the middle wire being adjusted in the meridian.

The instrument may thus be used for transit observations to determine the clock correction.

A list of suitable pairs of stars should be prepared beforehand, with the zenith distance and clock time of culmination of each star. There should be an interval of at least two or three minutes between the culmination of the two stars of a pair, in order to give time to read the micrometer and level, and to turn the instrument 180° for the other star. The interval between two *pairs* should be somewhat longer, as the telescope has also to be set at a new zenith distance.

The instrument being levelled and the stops so placed as to fix the position of the telescope in the meridian when turned either north or south, the telescope is set to the mean of the zenith distances of the two stars, turned in the direction of the star which culminates first, and clamped against the stop. Soon after the star enters the field, the observer bisects it with the micrometer wire, and keeps it bisected until the instant of culmination, then reads the micrometer and level, and turns the instrument against the other stop; and the second star is observed in the same manner.

113. Formula for the Latitude. — Suppose m to be the micrometer reading on the south star, m' that on the north star, and R the value of one revolution of the screw in seconds; then the measured value of $z - z'$ in seconds will be $R(m - m')$ or $R(m' - m)$, according as the micrometer reads *from* the zenith or *towards* the zenith.

Let n and s denote the readings of the north and south ends of the level at the time of observation of the south star, n' and s' at that of the north star, and d the value of one division of the level scale in seconds, then

$$d\frac{n-s}{2} \quad \text{and} \quad d\frac{n'-s'}{2}$$

will be the corresponding inclinations of the level to the horizon. Since any change of level which increases the apparent zenith distance of one star diminishes that of the other, the observed difference of zenith distance must be corrected by the *sum* of these inclinations, that is,

$$\frac{d}{2}(n - s + n' - s').$$

A correction is also required for the difference of refraction due to the slight difference of zenith distance of the two stars. Let r be the refraction for the south, and r' that for the north star; as they diminish the apparent zenith distance of both stars, their *difference*, $r - r'$, will be the correction to be added. Hence the corrected difference of zenith distance is

$$z - z' = R(m - m') + \frac{d}{2}[n + n' - (s + s')] + r - r',$$

and equation (8) becomes

$$\phi = \tfrac{1}{2}(\delta + \delta') + \frac{R}{2}(m - m') + \frac{d}{4}[n + n' - (s + s')] + \tfrac{1}{2}(r - r') \qquad (9)$$

If the observation on either star was made after its meridian passage, another correction must be added for the "reduction to the meridian." See Art. 123.

Example.

The following observations were made on two stars numbered 6426 and 6452 of the British Association Catalogue.

No.		*Microm. readings.*	*Level readings.* N	S
6426	S	19.065	33.0	34.8
6452	N	17.365	34.5	33.2
		$m - m' = 1.700$	67.5	68.0
				67.5
			$n + n' - (s + s') = -$	0.5

For the instrument used, $R = 43''.64$, $d = 1''$, and the micrometer read towards the zenith, hence we have by (9),

$$\begin{array}{rrl} & & \delta = 52^\circ\ 48'\ 47''.43 \\ 21.82\,(m - m') = & -\,0'37''.09 & \delta' = 32\ \ 40\ \ \ 9\ .03 \\ \frac{1}{4}[n + n' - (s + s')] = - & 0.13 & 2)\overline{85\ \ 28\ \ 55\ .46} \\ \frac{1}{2}(r - r') = - & 0.01 & \quad\ 42\ \ 44\ \ 27\ .73 \\ - & 37''.23 & \ldots\ \ -37\ .23 \\ & & \phi = 42^\circ\ 43'\ 50''.50 \end{array}$$

Latitude by the Prime Vertical Instrument. — Bessel's Method.

114. Formula for the Latitude. — When a body is on the prime vertical, its azimuth $Z = 90^\circ$, hence $\cos Z = 0$, and we have by (9) and (11) of Chapter I.,

$$\sin h = \frac{\sin \delta}{\sin \phi} \tag{10}$$

and

$$\cos P = \frac{\tan \delta}{\tan \phi} \tag{11}$$

whence

$$\tan \phi = \frac{\tan \delta}{\cos P} \tag{12}$$

From (12) we may find the latitude ϕ, when we know δ, the declination of the body, and P, its hour angle on the prime vertical. The latter must be found by observation.

115. Observation of Prime Vertical Transit. — A star whose declination is north and less than the latitude of the place will cross the meridian between the zenith and equator, and will cross the prime vertical at equal altitudes east and west of the meridian. If the transit instrument be adjusted with its axis north and south, so that the telescope revolves in the plane of the prime vertical, and if the passage of the star over the wires be observed at both

positions, the sidereal interval between the observations is double the star's hour angle on the prime vertical (Art. 47).

Hence, the value of P in equation (12) is half the observed sidereal interval converted into arc.

116. Adjustment in the Prime Vertical. — The transit instrument may be approximately adjusted in the prime vertical by placing it on a star at the instant of passing the prime vertical as nearly as the time can be ascertained. For this purpose, P may be computed from (11), using an approximate value of ϕ, and from P and the star's right ascension the required time can be found. The star's altitude at the same moment is given by equation (10).

If the instrument is accurately adjusted in the prime vertical, the mean of the observed sidereal times corrected for the error of the clock will be equal to the star's right ascension. If the two are not equal, their difference will measure the azimuth error of the instrument. Any such error, in either direction, makes the resulting latitude too great.

117. Practical Details. — The telescope should be reversed on its supports between the observation of the east and west transits, in order to eliminate the error of collimation. The level error should be well determined, and applied to the result.

The observations on each wire must be reduced to the middle wire by means of the equatorial intervals, or the latitude determined separately from the observations on each wire, and the mean of the results taken.

118. This method of finding the latitude was devised by the celebrated Prussian astronomer, Bessel, and in the accuracy of its results is second only to Talcott's method.

EXAMPLE.

α *Lyræ* on the Prime Vertical.

Observed sidereal time	west transit	$= 20^h\ 32^m\ 41^s.34$
	east transit	$= 16\ \ 32\ \ 49\ .46$
Sidereal interval,		$2)3\ \ 59\ \ 51\ .88$
Hour angle		$= 1^h\ 59^m\ 55^s.94$
		$= 29°\ 58'\ 59''.10$

$\delta = 38°\ 39'\ 55''.1$	tan	9.903176
$P = 29\ \ 58\ \ 59\ .1$	cos	9.937605
$\phi = 42°\ 43'\ 52''.6$	tan	9.965571

LATITUDE BY A SINGLE ALTITUDE AND THE CORRESPONDING TIME.

119. Formula for the Latitude. — Equation (8) of Chapter I., namely,

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos P \tag{13}$$

may be solved for ϕ as follows:

Assume m and M such that we have

$$\sin \delta = m \sin M \tag{14}$$

$$\cos \delta \cos P = m \cos M \tag{15}$$

whence
$$\tan M = \frac{\tan \delta}{\cos P} \tag{16}$$

The substitution of (14) and (15) reduces (13) to

$$\sin h = m \cos (\phi - M),$$

whence
$$\cos (\phi - M) = \frac{\sin h}{m} = \frac{\sin h \sin M}{\sin \delta},$$

and
$$\phi = M + \cos^{-1} \frac{\sin h \sin M}{\sin \delta} \tag{17}$$

In order to apply this formula in practice, the hour angle (P) and altitude (h) of a body whose place is known must be found by observation.

120. The Observations. — The altitude may be measured with the sextant, and the hour angle may be found by

noting the time at which the altitude was measured, and substituting it in (8) or (10) of Chapter II. The clock error must be accurately known, however, as any error in the observed time will affect the value of P.

But the hour angle may be found independently of the clock error, as explained in Art. 47, by observing the times of equal altitude east and west of the meridian. Half the difference of *sidereal* times of equal altitudes of a star, or half the difference of *apparent* times of those of the sun, is the hour angle expressed in time, the clock error being eliminated in taking the interval. The clock *rate* must be known, however, as it affects the observed interval.

If the hour angle is found by this method, the value of δ to be used in (16) and (17) should be the mean of the declinations for the two observed times.

Example.

Double altitudes of the sun's upper limb, and corresponding mean times. Bar. 30.6 in.; Ther. 35°.5.

	Double altitudes.
	56° 50′
	56 52
	56 58
	57 00
	57 02
	5)284 42
	56° 56′ 24″.0
Ind. cor. =	−1 27 .5
	2)56 54 56 .5
	28 27 28 .25
Ref. =	1 52 .79
	28 25 35 .46
Par. =	7 .87
	28 25 43 .33
Semi-d. =	16 13 .50
h =	28 9 29 .83

Observed times.	
23^h 22^m 06^s	
23 05	
26 02	
27 02	
28 03	
5)126 18	
23 25 15 .60	= obs'd time.
+33 .32	= clock cor.
23 25 48 .92	= mean time.
14 56 .60	= eq. of time.
23 40 45 .52	= app. time.
24	
− 0^h 19^m $14^s.48$	or
− 4° 48′ 37″.2	= hour angle.

$P = -\ \ 4° 48' 37''.2$	cos 9.998467		
$\delta = -18\ 56\ 59\ .7$	tan $-$ 9.535739		a. c. sin $-$ 0.488460
$M = -19\ 00\ 44$	tan $-$ 9.537272		sin $-$ 9.512931
		$h = 28° 9' 29''.8$	sin 9.673859
$\phi - M = 61° 44' 36''$			cos 9.675250
$\phi = 42° 43' 52''$			

121. Best Position for Latitude Observations. — To find the condition in which a small error in the measurement of the altitude will have the least effect on the result, differentiate equation (8) of Chapter I., with respect to h and ϕ; we find

$$\cos h\, dh = (\sin\delta\cos\phi - \cos\delta\cos P\sin\phi)\,d\phi.$$

This reduces, by (11) of Chapter I., to

$$\cos h\, dh = \cos h\cos Z\, d\phi,$$

whence
$$d\phi = \frac{dh}{\cos Z}.$$

Hence $d\phi$ is a minimum when $\cos Z$ is a maximum $= 1$, or $Z = 0°$ or $180°$, that is, the body is on the meridian.

It follows that altitudes measured for the purpose of finding the latitude should be taken when the body is on or near the meridian.

Latitude by Circum-meridian Altitudes.

122. This method consists in measuring several altitudes of a body just before and just after its meridian passage, applying to each a correction called the "reduction to the meridian," and taking their mean as the meridian altitude, from which the latitude may be found by one of the formulæ of Art. 110.

123. Reduction to the Meridian. — The *reduction to the meridian* is the difference between any observed altitude or

zenith distance, and the meridian altitude or zenith distance.

Let S, Fig. 14, be the place of a body when its altitude or zenith distance is measured, ST an arc of its diurnal

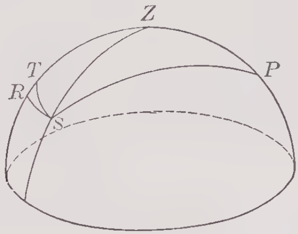

FIG. 14.

circle, and SR an arc parallel to the horizon; then by Art. 33,

$$\text{meridian zenith distance } ZT = \phi - \delta.$$

Let $\quad z = ZS = ZR =$ observed zenith distance;

and $\quad x = ZS - ZT = TR =$ reduction to meridian;

then we have

$$z = x + \phi - \delta,$$

and $$\cos z = \cos x \cos(\phi - \delta) - \sin x \sin(\phi - \delta).$$

But the observations being made near the meridian, say within 10 minutes of the time of meridian passage, x is very small, and we may put

$$\cos x = 1, \quad \sin x = x \sin 1'';$$

also we may replace $\cos z$ by $\sin h$, then

$$\sin h = \cos(\phi - \delta) - x \sin(\phi - \delta) \sin 1'' \qquad (18)$$

If now we substitute in equation (8), Chapter I.,

$$\cos P = 1 - 2 \sin^2 \tfrac{1}{2} P,$$

we have

$$\sin h = \sin\delta \sin\phi + \cos\delta \cos\phi - 2 \cos\delta \cos\phi \sin^2 \tfrac{1}{2} P,$$

or $$\sin h = \cos(\phi - \delta) - 2 \cos\delta \cos\phi \sin^2 \tfrac{1}{2} P \qquad (19)$$

Equating the second members of (18) and (19), we find

$$x \sin(\phi - \delta) \sin 1'' = 2 \cos\delta \cos\phi \sin^2 \tfrac{1}{2} P,$$

whence
$$x = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''} \cdot \frac{\cos\delta \cos\phi}{\sin(\phi - \delta)},$$

or
$$x = k \cdot \frac{\cos\delta \cos\phi}{\sin(\phi - \delta)} \tag{20}$$

in which x is expressed in *seconds*. This is the correction to be applied to the observed altitudes.

The values of
$$k = \frac{2 \sin^2 \frac{1}{2} P}{\sin 1''} \tag{21}$$

are computed for values of P to every second, and may be taken from Table III.

Since equation (20) involves the latitude, ϕ, it must be approximately known before this method can be used.

124. Correction for Rate of Clock. — If the clock has a sensible *rate* during the observations, it must be taken into account.

Let r = daily rate, positive when *losing;* then if P is the hour angle indicated by the clock, and P' the true hour angle, we have

$$\frac{P'}{P} = \frac{24^{\mathrm{h}}}{24^{\mathrm{h}} - r} = \frac{86400^{\mathrm{s}}}{86400^{\mathrm{s}} - r} = \frac{1}{1 - \frac{r}{86400}}.$$

Let
$$\left(\frac{P'}{P}\right)^2 = \left(\frac{1}{1 - \frac{r}{86400}}\right)^2 = n.$$

In (21) we should use $\sin \frac{1}{2} P'$ instead of $\sin \frac{1}{2} P$, but as P is always small,

$$\sin \tfrac{1}{2} P' : \sin \tfrac{1}{2} P = P' : P, \text{ nearly,}$$

and
$$\sin^2 \tfrac{1}{2} P' = \sin^2 \tfrac{1}{2} P \left(\frac{P'}{P}\right)^2 = n \sin^2 \tfrac{1}{2} P.$$

The factor k in equation (20) now becomes nk, and the value of $\log n$ is given in Table III.

Example.

Circum-meridian altitudes of the sun's upper limb and corresponding mean times. Bar. 29.5 in.; Ther. 52°.3.

Double alt.	*Mean times.*	*App. times.*	*H. ang.*	*k*
68° 27′ 00″	$23^h\,35^m\,26^s$	$23^h\,51^m\,33^s$	$8^m\,27^s$	140.18
28 00	36 37	52 44	7 16	103.67
29 00	37 50	53 57	6 03	71.86
30 00	39 24	55 31	4 29	39.46
30 40	43 38	59 45	0 15	0.12
29 00	48 38	24 4 45	4 45	44.30
28 00	50 32	6 39	6 39	86.82
27 00	52 05	8 12	8 12	132.01
25 00	53 37	9 44	9 44	185.99
24 00	54 45	10 52	10 52	231.81
10) 277 40				10) 1036.22
68° 27′ 46″.				103.62

```
   −1 37 .5  = index correction.
2) 68 26  8 .5
   34 13  4 .25 = observed altitude.
       1 23 .88 = refraction.
   34 11 40 .37
          7 .37 = parallax.
   34 11 47 .74
      16  9 .00 = semi-diameter.
   33 55 38 .74 = altitude center.
   90 00 00 .00
   56  4 21 .26 = zenith distance.
       1 29 .29 = x.
   56  2 51 .97 = z = meridian zenith distance.
  −13 19  2 .60 = δ.
   42 43 49 .37 = φ.   [By equation (2).]
```

Computation of x, equation (20).

$k =$	103.62	log	2.015452
$\delta =$	$-13^\circ\ 19'\ \ 2''.6$	cos	9.988160
$\phi =$	$42\ \ 43\ \ 50\ \ .0$	cos	9.866023
$\phi - \delta =$	$56\ \ \ 2\ \ 52\ \ .6$	a. c. sin	0.081181
$x =$	$89''.293$	log	1.950817

CHAPTER VIII.

FINDING THE AZIMUTH OF A GIVEN LINE.

125. The Meridian Line. — The intersection of the plane of the meridian at any place with that of the horizon is called the *meridian line.* The direction of the meridian line is determined by finding the azimuth of a distant terrestrial point, that is, the angle which a vertical plane through the point and the observer's eye makes with the plane of the meridian.

126. Azimuth determined by Observation. — The general method of determining azimuth by the observation of a heavenly body is to measure the difference of azimuth of the body and a fixed mark at a given instant, then compute the azimuth of the body at that instant; the azimuth of the mark will then be known. The difference of azimuth is measured directly by pointing the instrument to the mark and the star in succession, and reading the horizontal circle for each. A number of repetitions of the observation should be made to secure greater accuracy.

127. The accurate determination of azimuth requires the use of a theodolite with large circles which are read by microscopes. An approximate result may be found, however, with a smaller theodolite, an altitude and azimuth instrument, or the engineer's transit.

The terrestrial mark should be so far distant that it may be distinctly observed by the telescope when focused on a star. A convenient mark for night observations is formed by the light of a lantern shining through a narrow vertical slit in a screen placed in front of it.

Azimuth by Observing a Circumpolar Star at its Greatest Elongation.

128. Azimuth, Altitude, and Hour Angle at Elongation.— A star is said to be at its greatest elongation when its vertical circle, ZS, Fig. 1, is tangent to its diurnal circle, that is, perpendicular to its hour circle PS. Hence in this position the parallactic angle $S = 90°$, $\cos S = 0$, and $\sin S = 1$; whence we find by (6), (10), and (13) of Chapter I.,

$$\sin Z = \frac{\cos \delta}{\cos \phi} \tag{1}$$

$$\sin h = \frac{\sin \phi}{\sin \delta} \tag{2}$$

$$\cos P = \frac{\sin h \cos \delta}{\cos \phi} = \frac{\tan \phi}{\tan \delta} \tag{3}$$

The latitude of the station being known, equations (1) and (3) make known the star's azimuth and hour angle at its greatest elongation. From the latter, the sidereal time of elongation is found by equation (9) of Chapter II., and may be converted into mean time at the same instant. (Art. 43.)

129. Method of Observing. — The observation consists in measuring the difference of azimuth of the star at elongation and the fixed mark. Before the time of greatest elongation as found above, let the instrument be set up and carefully adjusted; then bisect the mark with the vertical wire, and read the horizontal circle. Now direct the telescope to the star, bisect it with the wire and follow it by means of the tangent-screw, keeping it bisected till it ceases to change its azimuth; then read the circle. The observation on the mark should now be repeated, and the mean of the readings on the mark compared with that on

the star; their difference is the difference of azimuth required. Knowing the azimuth of the star and the difference of azimuth of the star and mark, we know the azimuth of the mark, which fixes the direction of the meridian line.

130. Suitable Circumpolar Stars. — *Polaris* (α Ursæ Minoris) being of the second magnitude and close to the pole, is a convenient star to observe for azimuth. The stars δ and λ of the same constellation, and 51 Cephei, are also sometimes used; but as they are of smaller magnitudes, they can only be observed with the larger instruments. The Nautical Almanac gives the places of these four stars for every day of the year.

Example.

Polaris at greatest eastern elongation. Star east of mark.

$\phi = 42° 43' 53''$	tan 9.965573	cos 9.866017
$\delta = 88\ 43\ 13$	tan 1.650935	cos 8.348957
$P = 88\ 49\ 03$	cos 8.314638	sin Z, 8.482940
$= -5^h 55^m 16^s$		$Z = 1° 44' 32''$
$\alpha = 25\ 19\ 16$		

$19^h 24^m$ = sidereal time of greatest elongation.

Reading on star at elongation	$= 14° 24' 52''$
Mean reading on mark	$= 6\ 13\ 25$
Difference of azimuth	$= 8\ 11\ 27$
Azimuth of star (east of north)	$= 1\ 44\ 32$
Azimuth of mark (west of north)	$= 6° 26' 55''$

Azimuth by Observing a Body at a Given Instant.

131. The observation on the body consists in taking a series of azimuth readings and the corresponding times. The mean of the readings compared with the reading on

the mark gives their difference of azimuth, and the mean of the observed times corrected for clock error is the instant at which the body's azimuth is required. Its hour angle at this instant may be found by equation (8) or (10) of Chapter II.; then, its declination and the latitude of the place being known, we have, equations (35), (36) and (37), Chapter I.,

$$\tan M = \frac{\tan \delta}{\cos P} \tag{4}$$

$$\tan Z' = \frac{\cos M \tan P}{\sin (\phi - M)} \tag{5}$$

$$\tan h = \frac{\cos Z'}{\tan (\phi - M)} \tag{6}$$

131 a. Observations on the Sun.—In the case of the sun, if an equal number of observations be made on the east and west limbs, respectively, the mean of the readings thus taken will be the reading for the center. The axis of the instrument should be reversed between the observations on the two limbs.

If, however, observations be made on one limb only, the azimuth of the center, Z', must be corrected by the difference of azimuth of the limb and the center.

Let AZ be the vertical circle tangent to the limb at A, and draw the radius CA, then

$$CZ = z = \text{zenith distance of center.}$$

Let also

$$CA = s = \text{sun's apparent semi-diameter.}$$

$$AZC = \Delta = \text{difference of azimuth of center and limb.}$$

In the spherical triangle ZAC, right-angled at A, we have

$$\sin s = \sin z \sin \Delta = \cos h \sin \Delta,$$

whence $\sin \Delta = \sin s \sec h;$

or, practically, $\Delta = s \sec h$ (7)

EXAMPLE.

Observations on the sun's east limb with the engineer's transit. Sun west of mark.

Observed sidereal times.	*Readings.*
$16^h\ 1^m\ 12^s.9$	75° 00′
1 55.2	10
2 42.0	20
3 24.2	30
4 9.8	40
4 51.8	50
6 22.0	76 10
7 9.0	20
7 53.0	30
8 38.7	40
10)48 18.6	10)758 10
$16^h\ 4^m\ 49^s.86$	75° 49′ 00″
Clock fast 23.22	

$16^h\ 4^m\ 26^s.64$ = sidereal time of observation.

13 4 54.29 = sun's R. A.

$2^h\ 59^m\ 32^s.35$ = sun's hour angle

$= P =$ 44° 53′ 05″	cos 9.850367	tan 9.998240
$\delta = -6$ 54 49	tan − 9.083709	
$M = -9$ 42 41	tan − 9.233342	cos 9.993731
$\phi =$ 42 43 53		
$\phi - M =$ 52° 26′ 34″	tan 0.114122	a. c. sin 0.100867
$Z' =$ 51° 04′ 40″	cos 9.798143	tan 0.092838
$h =$ 25 47 04	tan 9.684021	sec 0.045546
$s =$ 16′ 04″ = 964″		log 2.984077
$\Delta =$ 17′ 51″ = 1071″		log 3.029623

Mean reading on E. limb	$= 75° \, 49' \, 00''$
Mean reading on mark	$= 29 \;\; 37 \;\; 45$
Difference of azimuth	$= 46 \;\; 11 \;\; 15$
Azimuth E. limb $= Z' - \Delta$	$= 50 \;\; 46 \;\; 49$
Azimuth of mark (W. of S.)	$= \;\; 4° \, 35' \, 34''$

Azimuth by Observing a Body at a Given Altitude.

132. If the *altitudes* be taken, instead of the times, corresponding to a series of azimuth readings on the body, we have, equation (27), Chapter I.,

$$\sin \tfrac{1}{2} Z = \sqrt{\frac{\sin \frac{1}{2} (z + \phi - \delta) \cos \frac{1}{2} (z + \phi + \delta)}{\cos \phi \sin z}} \qquad (8)$$

in which z is found from the observed altitudes, and Z is the azimuth of the body from the north point.

Azimuth by Observing a Body at Equal Altitudes.

133. **Equal Altitudes of a Star.** — Since a star is a fixed point, and since the diurnal motion is uniform, equal altitudes of any star correspond to equal azimuths. Hence if a star be observed at equal altitudes east and west of the meridian, and if r and r' denote the observed azimuth readings at equal altitudes, we shall have

$$\frac{r + r'}{2} = \text{azimuth reading for the meridian.}$$

The azimuth of the fixed mark will therefore be the difference between the reading on the mark and the mean of the two readings on the star.

It is here assumed that the readings of the horizontal circle are continuous between the observations.

134. Equal Altitudes of the Sun.—If the sun be observed at equal altitudes, the mean of the two readings will require a correction on account of its change of declination during the interval.

Take equations (5) and (9) of Chapter I., and replace Z by its value $180° - Z'$; they become

$$\cos h \sin Z' = \cos \delta \sin P \tag{9}$$

$$\sin \delta = \sin \phi \sin h - \cos \phi \cos h \cos Z' \tag{10}$$

To find what effect the sun's change of declination produces in its azimuth, differentiate (10) in respect to δ and Z' as variables; we find, by (9),

$$\frac{dZ'}{d\delta} = \frac{\cos \delta}{\cos \phi \cos h \sin Z'} = \frac{1}{\cos \phi \sin P} \tag{11}$$

The times of the two observations having been noted, let

$\Delta = \frac{1}{2}$ sun's change of declination in the interval, and

$x =$ correction required by mean of readings.

Then we may put $\frac{x}{\Delta}$ for $\frac{dZ'}{d\delta}$ in equation (11), whence

$$x = \frac{\Delta}{\cos \phi \sin P} \tag{12}$$

The value of Δ may be found from the hourly change of declination given in the Nautical Almanac, and the hour angle P is half the interval of apparent time between the observations, converted into arc. (Art. 44.)

The *sign* of the correction x is determined by the consideration that if the sun's declination is increasing, the mean of the readings at equal altitude will lie *west* of the meridian, but if decreasing, *east* of the meridian.

CHAPTER IX.

FIGURE AND DIMENSIONS OF THE EARTH.

FORMULÆ FOR THE SPHEROID.

135. The investigation of the formulæ which determine the figure and dimensions of the earth belongs rather to Geodesy than to Astronomy, but their importance in connection with some of the applications of astronomy warrants a brief discussion of the subject in this place.

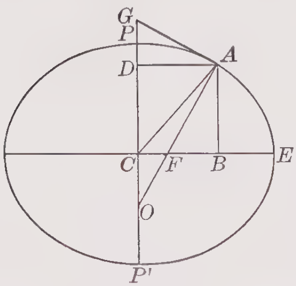

FIG. 15.

136. Notation. — The earth being regarded as an oblate spheroid, let PEP', Fig. 15, be a section through its axis PP' and the point A, the place of the observer. Let P be the north pole, P' the south pole, and E a point on the equator. Draw the tangent AG and the normal AO at the point A, then OA produced will meet the observer's zenith, and the radius CA produced will meet the celestial sphere in the geocentric zenith.

Let $CE = a =$ equatorial radius,

$CP = b =$ polar radius,

$CA = \rho =$ radius at given place,

$AFB = \phi =$ latitude at given place,

$ACB = \phi' =$ geocentric latitude,

$AG = t \; =$ tangent ending at minor axis,

$AO = N =$ normal ending at minor axis,

$AF = n =$ normal ending at major axis,

$FB = s =$ subnormal on major axis.

Let $CB = x$, $AB = y$, the rectangular co-ordinates of the given place, the origin being at the earth's center.

In the problems which follow, the latitude, ϕ, will be assumed to be known.

137. Fundamental Relations.—The right-angled triangles AGO, ADO, and ADG are similar to the triangle ABF, and to each other.

The right triangle ABC gives

$$\sin \phi' = \frac{y}{\rho} \tag{1}$$

$$\cos \phi' = \frac{x}{\rho} \tag{2}$$

$$\tan \phi' = \frac{y}{x} \tag{3}$$

The similar triangles ABF, ADG, give

$$\sin \phi = \frac{y}{n} = \frac{x}{t} \tag{4}$$

The similar triangles ABF, ADO, give

$$\cos \phi = \frac{s}{n} = \frac{x}{N} \tag{5}$$

The similar triangles ABF, AGO, give

$$\tan \phi = \frac{y}{s} = \frac{N}{t} \tag{6}$$

138. Geocentric Latitude. — The general formula for the subnormal of the ellipse is

$$s = (1 - e^2)x,$$

e being the eccentricity. Hence by (3),

$$\tan \phi' = \frac{y}{x} = \frac{y}{s}(1 - e^2),$$

or, by (6),

$$\tan \phi' = (1 - e^2) \tan \phi \tag{7}$$

The angle $CAO = AFB - ACB = \phi - \phi'$, and is called the *angle of the vertical*, or the *reduction of latitude.*

139. Co-ordinates of the Given Place. — The equation of the meridian ellipse PEP' is

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \tag{8}$$

or since

$$b^2 = a^2(1 - e^2) \tag{9}$$

$$y^2 = (1 - e^2)(a^2 - x^2) \tag{10}$$

But comparing (3) and (7) we have

$$y = x(1 - e^2) \tan \phi \tag{11}$$

Squaring (11), and equating with (10), we find

$$a^2 - x^2 = x^2(1 - e^2) \tan^2 \phi.$$

Hence $$x^2 = \frac{a^2}{1 + (1 - e^2)\tan^2 \phi} = \frac{a^2 \cos^2 \phi}{\cos^2 \phi + \sin^2 \phi - e^2 \sin^2 \phi},$$

and

$$x = \frac{a \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}} \tag{12}$$

and by (11),

$$y = \frac{a(1 - e^2) \sin \phi}{\sqrt{1 - e^2 \sin^2 \phi}} \tag{13}$$

If we make

$$e \sin \phi = \sin \chi \tag{14}$$

(12) and (13) may be written in the form.

$$x = a \sec \chi \cos \phi \tag{15}$$

$$y = a(1 - e^2) \sec \chi \sin \phi \tag{16}$$

which are adapted to logarithmic computation.

140. Tangent and Normal. — Equation (5) gives

$$N = \frac{x}{\cos \phi},$$

whence we have by (12),

$$\text{Normal ending at minor axis} = N = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \tag{17}$$

Equation (6) gives

$$\text{Tangent ending at minor axis} = t = N \cot \phi \tag{18}$$

From equation (5) we find

$$\frac{n}{N} = \frac{s}{x} = 1 - e^2, \text{ hence}$$

$$\text{Normal ending at major axis} = n = N(1 - e^2) \tag{19}$$

141. Radius of the Earth at the Given Place. — From Fig. 15, by (15) and (16),

$$\rho^2 = x^2 + y^2 = a^2 \sec^2 \chi [\cos^2 \phi + (1 - e^2)^2 \sin^2 \phi],$$

hence
$$\rho = a \sec \chi [1 + (e^4 - 2e^2) \sin^2 \phi]^{\frac{1}{2}}$$
$$= a \sec \chi [1 - e^2 (2 - e^2) \sin^2 \phi]^{\frac{1}{2}} \tag{20}$$

142. Radius of Curvature of the Meridian. — Let

R_m = radius of curvature of the meridian at any point A.

The expression for the radius of curvature of the ellipse is

$$R_m = \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4} = \frac{\left(\frac{a^2}{b^2} y^2 + \frac{b^2}{a^2} x^2\right)^{\frac{3}{2}}}{ab},$$

and by (8) and (9) we have

$$\frac{a^2}{b^2}y^2 = a^2 - x^2, \quad \frac{b^2}{a^2}x^2 = (1-e^2)x^2, \quad ab = a^2(1-e^2)^{\frac{1}{2}};$$

whence
$$R_m = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{a^2(1-e^2)^{\frac{1}{2}}}.$$

But from (12),

$$a^2 - e^2x^2 = a^2 - \frac{a^2e^2\cos^2\phi}{1-e^2\sin^2\phi} = \frac{a^2 - a^2e^2}{1-e^2\sin^2\phi},$$

hence

$$R_m = \frac{1}{a^2(1-e^2)^{\frac{1}{2}}} \cdot \frac{a^3(1-e^2)^{\frac{3}{2}}}{(1-e^2\sin^2\phi)^{\frac{3}{2}}} = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}} \qquad (21)$$

and from (17),

$$R_m = \frac{1-e^2}{a^2}N^3 \qquad (22)$$

143. Radius of Curvature of a Prime Vertical Section. — Since the earth is regarded as an ellipsoid of revolution, the direction of gravity, that is the vertical line, intersects the earth's axis; and for consecutive points of the prime vertical section it intersects it in the same point, which point is therefore the center of curvature of this section; hence its radius of curvature is the normal ending at the minor axis, that is, equation (17),

$$N = \frac{a}{\sqrt{1-e^2\sin^2\phi}} \qquad (23)$$

144. Radius of a Parallel of Latitude. — Let R_p = radius of the parallel of the given point A, then, Fig. 15,

$$R_p = AD = x,$$

hence by (12),

$$R_p = \frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}} \qquad (24)$$

and from (17),

$$R_p = N\cos\phi \qquad (25)$$

145. Length of a Degree of the Meridian. — Let D_m be the length of 1° of the meridian at the given place, then, by (21),

$$D_m = \frac{\pi R_m}{180} = \frac{\pi}{180} \cdot \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}};$$

or developing the denominator,

$$D_m = \frac{\pi}{180} a(1-e^2)\left(1+\tfrac{3}{2}e^2\sin^2\phi+\tfrac{15}{8}e^4\sin^4\phi+\text{etc.}\right) \quad (26)$$

If we neglect the fourth and higher powers of e, we have approximately,

$$D_m = \frac{\pi}{180} a(1-e^2)\left(1+\tfrac{3}{2}e^2\sin^2\phi\right),$$

or, if we put

$$\left.\begin{aligned} h &= \frac{\pi}{180} a(1-e^2) \\ k &= \tfrac{3}{2}e^2 h \end{aligned}\right\} \quad (27)$$

we have $\qquad D_m = h + k\sin^2\phi \qquad (28)$

At the equator, $D_m = h$; and at the pole, $D_m = h + k$.

146. If in (26) we substitute the values

$$\sin^2\phi = \tfrac{1}{2}(1-\cos 2\phi),$$
$$\sin^4\phi = \tfrac{1}{8}(3 - 4\cos 2\phi + \cos 4\phi), \text{ etc.},$$

it takes the form

$$D_m = \frac{\pi a}{180}(A + B\cos 2\phi + C\cos 4\phi + \text{etc.}) \quad (29)$$

in which

$$A = 1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4;\ B = -\tfrac{3}{4}e^2 - \tfrac{3}{16}e^4;\ C = \tfrac{15}{64}e^4.$$

147. Length of a Degree of a Parallel of Latitude. — Let D_p be the length of one degree of the parallel of latitude ϕ.

Then, by (24),

$$D_p = \frac{\pi R_p}{180} = \frac{\pi}{180} \cdot \frac{a \cos \phi}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}}$$

$$= \frac{\pi a}{180} \cos \phi (1 + \tfrac{1}{2} e^2 \sin^2 \phi + \tfrac{3}{8} e^4 \sin^4 \phi + \text{etc.}) \qquad (30)$$

By Trigonometry,

$$\cos \phi \sin^2 \phi = \tfrac{1}{4} (\cos \phi - \cos 3 \phi),$$

$$\cos \phi \sin^4 \phi = \tfrac{1}{16} (2 \cos \phi - 3 \cos 3 \phi + \cos 5 \phi), \text{ etc.},$$

the substitution of which reduces (30) to the form

$$D_p = \frac{\pi a}{180} (A' \cos \phi + B' \cos 3 \phi + C' \cos 5 \phi + \text{etc.}) \quad (31)$$

in which

$$A' = 1 + \tfrac{1}{8} e^2 + \tfrac{3}{64} e^4; \;\; B' = - \tfrac{1}{8} e^2 - \tfrac{9}{128} e^4; \;\; C' = \tfrac{3}{128} e^4.$$

148. Ellipticity of the Earth. — Let

$$c = \text{ellipticity} = \frac{a - b}{a} = 1 - \frac{b}{a} = 1 - \sqrt{1 - e^2},$$

then $\qquad (1 - c)^2 = 1 - e^2, \;$ hence $\; e^2 = 2c - c^2.$

If we neglect c^2, which is very small, we have, by (27),

$$c = \tfrac{1}{2} e^2 = \tfrac{1}{3} \frac{k}{h} \qquad (32)$$

Elements of the Spheroid as Determined by Measurement.

149. We have next to show how the values of the constants a and e, which enter into the preceding formulæ, may be found from the actual measurement of arcs of the meridian.

150. Triangulation. — The following is the method of determining the exact length of an arc of the meridian: A base line BL, Fig. 16, is selected on a level plain, several

miles in extent, and its length carefully measured by the most refined and accurate methods. A number of stations, A, B, C, etc., are also chosen as the vertices of a series of triangles extending in a north and south direction, these stations being so situated that in any one triangle the vertex of either angle can be seen from the other two. All the angles of each triangle are then carefully measured, and also the inclination of its sides to the true meridian. The angles being cleared from spherical excess, the sides of all the triangles are computed, beginning with ABL, of which the side BL was measured. Near the other extremity of the chain of triangles, a "verification base" is also measured, and unless there is a close agreement between its computed and measured length, the whole process is repeated.

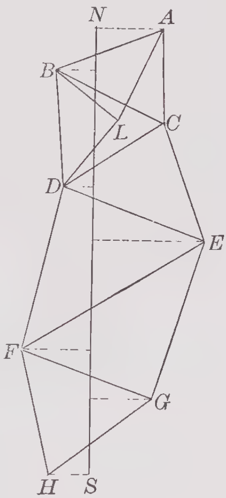

FIG. 16.

The length of each side multiplied by the cosine of its inclination to the meridian, gives its projection on the meridian, and the sum of the projections of AB, BC, CD, ..., GH, gives the length of the meridian line NS.

151. Length of a Degree by Measurement. — The latitudes of the extreme stations having been found by observation, let

$$\phi = \text{their half sum},$$

$$d = \text{their difference},$$

$$l = \text{length of measured line } NS,$$

$$D_m = \text{length of } 1^\circ \text{ of the meridian}.$$

Then $$d : 1^\circ = l : D_m,$$

whence $$D_m = \frac{l}{d}.$$

Let the values of ϕ and D_m be substituted in equation (28). Two such measurements on different arcs will furnish two equations, from which the values of h and k can be found. It will be better, however, to increase the number of equations, and solve them by the method of Least Squares.

The values of h and k being thus obtained, e and a become known from equations (27), b from (9), and c from (32).

152. Bessel's Elements. — The elements of the terrestrial meridian generally adopted, until within a few years, are those deduced by Bessel, and published by him in 1841. They were obtained by a discussion by the method of Least Squares, of the results of measurements of meridian arcs made previous to that time, and are as follows:

$$a = 20{,}923{,}597 \text{ feet},$$
$$b = 20{,}853{,}654 \text{ feet},$$
$$e^2 = 0.00667437,$$
$$c = \frac{1}{299.15}.$$

153. Clarke's Elements. — In 1866, Col. A. R. Clarke, of the British Ordnance Survey, published the results of the discussion of measurements embracing a longer arc of the meridian, and involving more precise methods than those which formed Bessel's data. They are generally considered more accurate than Bessel's elements, and were adopted in 1880 by the United States Coast and Geodetic

Survey as the basis of its geodetical computations. They are as follows:

$$a = 20{,}926{,}062 \text{ feet},$$
$$b = 20{,}855{,}121 \text{ feet},$$
$$e^2 = 0.006768,$$
$$c = \frac{1}{294.98}.$$

From these values we have by equations (29) and (31), for any latitude, ϕ,

$$D_m = 364609.87 - 1857.14 \cos 2\phi + 3.94 \cos 4\phi;$$
$$D_p = 365538.48 \cos \phi - 310.17 \cos 3\phi + 0.39 \cos 5\phi;$$

which are expressed in feet.

154. In 1880, CLARKE published a new discussion of the subject, giving as a result the following elements of the terrestrial spheroid:

$$a = 20{,}926{,}202 \text{ feet},$$
$$b = 20{,}854{,}895 \text{ feet},$$
$$c = \frac{1}{293.465}.$$

These values give, in feet,

$$D_m = 364609.12 - 1866.72 \cos 2\phi + 3.98 \cos 4\phi;$$
$$D_p = 365542.52 \cos \phi - 311.80 \cos 3\phi + 0.40 \cos 5\phi.$$

CLARKE'S *Geodesy*, p. 322.

We shall conclude this chapter with two or three problems of the higher geodesy, the solution of which involves the application of some of the foregoing formulæ.

THE POLYCONIC PROJECTION.

155. The method of representing a portion of the earth's surface on a plane which is called the *Polyconic projection* supposes each parallel of latitude to be developed on a cone having the parallel for its base, and its vertex at the point

where a tangent to the meridian at its intersection with the parallel cuts the earth's axis produced. By this method, the degrees of the parallel preserve their true length, and the distortion of area is less than in any other mode of projection.

Another important advantage of this system over others is that by a simple and convenient mode of construction a map is produced on which the meridians and parallels of latitude intersect at right angles.

156. Angle at Vertex. — Suppose AB, Fig. 17, to be an arc of the parallel to be developed, PA and PB the meridians through its extremities, V the vertex of the cone tangent at AB. We have by (25), for the radius of the parallel,

$$R_p = N \cos \phi \qquad (33)$$

and by (18), for the radius of the developed parallel, or side of the tangent cone,

$$t = N \cot \phi \qquad (34)$$

Let α = angle at the center of the parallel subtended by the arc AB; and θ = angle at the vertex of the cone subtended by the same arc. Since the number of degrees, or of seconds, in arcs of the same length are inversely as their radii, we shall have

$$\frac{\theta}{\alpha} = \frac{R_p}{t} = \sin \phi,$$

whence $\theta = \alpha \sin \phi \qquad (35)$

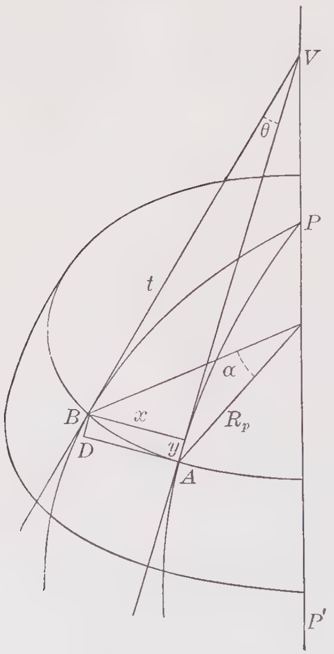

Fig. 17.

157. Co-ordinates of Curvature. — Drawing the lines AD perpendicular, and BD parallel, to AV, these will be the co-ordinates which determine the point B referred to A as the origin. Denoting them by x and y, we have

$$x = t \sin \theta,$$

$$y = t - t \cos \theta = t \operatorname{vers} \theta;$$

which may be put in the form

$$x = 2t \sin \tfrac{1}{2} \theta \cos \tfrac{1}{2} \theta \tag{36}$$

$$y = 2t \sin^2 \tfrac{1}{2} \theta = x \tan \tfrac{1}{2} \theta \tag{37}$$

For surfaces of small extent we may consider the arc AB to coincide with its chord, and since the angle between a tangent and chord is half the angle at the center subtended by the same arc, we have, the length of AB being αD_p,

$$x = \alpha D_p \cos \tfrac{1}{2} \theta \tag{38}$$

$$y = \alpha D_p \sin \tfrac{1}{2} \theta = x \tan \tfrac{1}{2} \theta \tag{39}$$

158. Construction. — The graphical construction consists in laying down by means of their co-ordinates the points of intersection of the parallel with each of the meridians of the map, the origin being the point of intersection of the parallel with the middle meridian. Drawing a right line perpendicular to the middle meridian at this point, and laying off the abscissas of the points of intersection on this line, and the ordinates perpendicular to it, a continuous curve drawn through the points thus determined will represent the developed parallel.

The distances between the parallels being laid off on the middle meridian, the same method of construction can be applied to each parallel in succession, and lines joining the corresponding points of the successive parallels will be the meridians of the map.

The values of the co-ordinates of curvature, x and y, may be taken from Projection Tables. The *Report of the Super-*

intendent of the United States Coast and Geodetic Survey for 1884, Appendix 6, contains a table computed on the basis of the CLARKE spheroid of 1866, giving the values of the co-ordinates in meters for each parallel of latitude, and for differences of longitude up to 30° on each side of the middle meridian.

SPHERICAL EXCESS OF TRIANGLES ON THE EARTH'S SURFACE.

159. It is shown in geometry that the sum of the angles of a spherical triangle is greater than two right angles, and that the excess of this sum over two right angles is to eight right angles as the area of the triangle is to the whole surface of the sphere.

Let $r =$ radius of the sphere,

$T =$ area of triangle,

$\epsilon =$ spherical excess;

then
$$\frac{\epsilon}{4\pi} = \frac{T}{4\pi r^2}, \text{ whence } \epsilon = \frac{T}{r^2};$$

or if ϵ is expressed in seconds,
$$\epsilon \sin 1'' = \frac{T}{r^2}.$$

In finding the *area* of a triangle on the earth's surface, we may regard it as a plane triangle; and if b and c are two of its sides, and A the included angle, we have
$$T = \tfrac{1}{2} bc \sin A,$$

and
$$\epsilon = \frac{bc \sin A}{2r^2 \sin 1''} \tag{40}$$

As r is not constant for the spheroid, we may take it as the mean of the radii of curvature of the meridian and prime vertical sections through the center of the triangle. That is,
$$r = \tfrac{1}{2}(R_m + N).$$

We have by (21),

$$R_m = a\,(1-e^2)\,(1-e^2\sin^2\phi)^{-\frac{3}{2}}$$
$$= a\,(1-e^2)\,(1+\tfrac{3}{2}\,e^2\sin^2\phi) = a\,(1+\tfrac{3}{2}\,e^2\sin^2\phi - e^2),$$

neglecting the fourth and higher powers of e. Similarly, by (23),

$$N = a\,(1-e^2\sin^2\phi)^{-\frac{1}{2}} = a\,(1+\tfrac{1}{2}\,e^2\sin^2\phi);$$

hence

$$r = \tfrac{1}{2}\,(R_m+N) = a\,(1-\tfrac{1}{2}\,e^2+e^2\sin^2\phi)$$
$$= a\,[1-\tfrac{1}{2}\,e^2+\tfrac{1}{2}\,e^2\,(1-\cos 2\phi)] = a\,(1-\tfrac{1}{2}\,e^2\cos 2\phi).$$

Hence for the spheroidal triangle we have, by (40),

$$\epsilon = \frac{bc\sin A}{2a^2\,(1-\frac{1}{2}\,e^2\cos 2\phi)^2\sin 1''} = mbc\sin A \qquad (41)$$

in which

$$m = \frac{1}{2a^2\,(1-\frac{1}{2}\,e^2\cos 2\phi)^2\sin 1''} \qquad (42)$$

Taking as the values of a and e those of the CLARKE spheroid of 1866, and adopting the meter as the unit of length, the values of $\log m$ have been computed for different values of ϕ, as in the annexed table.

ϕ	$\log m$	ϕ	$\log m$
36°	1.40491	46°	1.40390
37	81	47	80
38	71	48	69
39	61	49	59
40	51	50	49
41	41	51	39
42	31	52	29
43	20	53	19
44	10	54	09
45	1.40400	55	1.40299

Geodetic Determination of Latitudes, Longitudes, and Azimuths.

160. Let P, Fig. 18, be the pole of the earth, M and M' two points on its surface, and suppose the latitude and longitude of M are known, and that there are measured the length of the line MM' and the angle made by MM' with the meridian PM. The problem is to find the latitude and longitude of M', and the angle made by MM' with the meridian PM'. The following solution of the problem is essentially the same as the simplest of several methods given by Puissant in his *Traité de Géodésie*, Tome I.

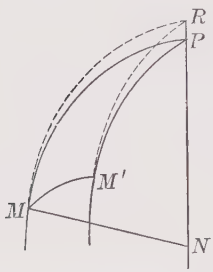

Fig. 18.

161. Notation. — Let ϕ denote the latitude of M, ϕ' that of M'; P the longitude of M, P' that of M'; Z the azimuth or bearing of MM' from the meridian at M, and Z' that at M'. The quantities ϕ, P, and Z are given, and ϕ', P', Z' are required.

The azimuths Z and Z' must be measured in the same direction from the meridian; suppose them reckoned from the south towards the east, then we have from the figure, $Z = 180° - M$, and $Z' = 180° + M'$.

Let PN be the polar axis, MN the normal at the point M, and R the pole of a sphere whose radius is MN, denoted by N. If we denote by K the measured length of the arc MM', and by u the same arc expressed in seconds, then considering MM' as a small arc of a great circle of the sphere whose radius is N, we have

$$\sin u = \frac{K}{N},$$

or by (17),
$$u'' = \frac{K}{N \sin 1''} = \frac{K(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}}{a \sin 1''} \tag{43}$$

162. Latitude on the Sphere. — In the spherical triangle RMM',

$$RM = 90° - \phi, \quad RM' = 90° - \phi', \quad M = 180° - Z;$$

hence

$$\cos RM' = \sin \phi' = \sin \phi \cos u - \cos \phi \sin u \cos Z.$$

Let the difference of latitude $\phi' - \phi = x$, x being expressed in seconds, then

$$\sin \phi' = \sin (\phi + x) = \sin \phi \cos x + \cos \phi \sin x,$$

whence

$$\sin \phi \cos x + \cos \phi \sin x = \sin \phi \cos u - \cos \phi \sin u \cos Z.$$

and $$\sin x = -\sin u \cos Z + \tan \phi (\cos u - \cos x) \qquad (44)$$

Since x and u are small arcs expressed in seconds, we may put $\sin x = x \sin 1''$, $\sin u = u \sin 1''$; and since

$$\cos x = (1 - \sin^2 x)^{\frac{1}{2}} = 1 - \tfrac{1}{2} \sin^2 x - \tfrac{1}{8} \sin^4 x, \text{ etc.},$$

we may also put

$$\cos x = 1 - \tfrac{1}{2} x^2 \sin^2 1'', \quad \cos u = 1 - \tfrac{1}{2} u^2 \sin^2 1'';$$

then $$\cos u - \cos x = -\tfrac{1}{2} (u^2 - x^2) \sin^2 1'',$$

and by (44),

$$x'' = -u'' \cos Z - \tfrac{1}{2} (u''^2 - x''^2) \sin 1'' \tan \phi \qquad (45)$$

We have for the first approximation,

$$x'' = -u'' \cos Z,$$

which substituted in the last term of (45), gives

$$x'' = -u'' \cos Z - \tfrac{1}{2} u''^2 \sin 1'' (1 - \cos^2 Z) \tan \phi,$$

and $$\phi' = \phi - u'' \cos Z - \tfrac{1}{2} u''^2 \sin 1'' \sin^2 Z \tan \phi \qquad (46)$$

163. Latitude on the Spheroid. — This value of ϕ' is the latitude of M' on the sphere, and is not the same as the latitude of M' considered as a point of the spheroid. To find the correction for the latitude, we observe that as in different circles, arcs of the same number of degrees are proportional to their radii, the number of degrees, or of

seconds, in equal arcs will be *inversely* as their radii. The radius of the sphere is

$$N = \frac{a}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}},$$

and that of the spheroid at the point M is, by (21),

$$R = \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}} = N \frac{1 - e^2}{1 - e^2 \sin^2 \phi}.$$

Let $z =$ difference of latitude, $(\phi' - \phi)$, on the spheroid, then $z : x = N : R$. Hence

$$z = x\,\frac{1 - e^2 \sin^2 \phi}{1 - e^2} = x\,[1 + (e^2 + e^4 + \text{etc.}) \cos^2 \phi],$$

or, $\quad z'' = x''\,(1 + e^2 \cos^2 \phi)$, very nearly.

Hence we have, by (46),

$$\phi' = \phi - (1 + e^2 \cos^2 \phi)\,(u'' \cos Z + \tfrac{1}{2} u''^2 \sin 1'' \sin^2 Z \tan \phi) \quad (47)$$

164. Longitude. — To find P', we observe that in the triangle RMM', the angle MRM' is the difference of longitude $P' - P$, hence we have

$$\sin (P' - P) : \sin M = \sin u : \cos \phi',$$

or, since $P' - P$ and u are very small, and $M = 180° - Z$,

$$(P' - P)'' \cos \phi' = u'' \sin Z,$$

whence

$$P' = P + \frac{u'' \sin Z}{\cos \phi'} \quad (48)$$

165. Azimuth. — To find Z', we have in the same triangle, by Napier's Analogies,

$$\tan \tfrac{1}{2}\,(M + M') = \cot \tfrac{1}{2}\,(P' - P)\,\frac{\cos \frac{1}{2}\,(\phi - \phi')}{\sin \frac{1}{2}\,(\phi + \phi')},$$

or, $$\cot \tfrac{1}{2}\,(M + M') = \tan\,[90° - \tfrac{1}{2}\,(M + M')]$$

$$= \tan \tfrac{1}{2}\,(P' - P)\,\frac{\sin \frac{1}{2}\,(\phi + \phi')}{\cos \frac{1}{2}\,(\phi - \phi')}.$$

But $90° - \frac{1}{2}(M+M')$ and $\frac{1}{2}(P'-P)$ are very small angles, hence we may put for their tangents the angles themselves; we also have $M+M' = Z'-Z$, and since $\phi - \phi'$ is very small, we may put $\cos\frac{1}{2}(\phi-\phi') = 1$. Substituting in the last equation, and multiplying by 2, we obtain

$$180° + Z - Z' = (P'-P)\sin\tfrac{1}{2}(\phi+\phi') \tag{49}$$

or by (48),

$$Z' = 180° + Z - \frac{u''\sin Z}{\cos\phi'}\sin\tfrac{1}{2}(\phi+\phi') \tag{50}$$

166. The logarithms of the factors $\dfrac{1}{N\sin 1''}$ and $1+e^2\cos^2\phi$, which enter equations (43) and (47), are computed for a series of values of ϕ and arranged in a table. Such a table, extending from 20° to 50° of latitude, may be found in the collection of Tables and Formulæ forming No. 12 of the *Professional Papers of the Corps of Engineers, U. S. A.* From the same work the following example is taken.

Data.

$K = 53644$ yards,	$P = 84°\,42'\,22''.19$,
$\phi = 45°\,39'\,13''.89$,	$Z = 159\;\;20\;\;13\;.62$.

1. To find ϕ' by (47):

$\log K$	4.7295212	$\log\frac{1}{2}\sin 1''$	4.38454
$\log\dfrac{1}{N\sin 1''}$	8.4701676	$2\log\sin Z$	9.09522
$\log u''$	3.1996888	$2\log u''$	6.39936
$\log(1+e^2\cos^2\phi)$	0.0014140		0.00141
$\log\cos Z$	$-$9.9711240	$\log\tan\phi$	0.00991
1st term $= -1486''.71$	$-$3.1722268	2d term $= 0''.77$	9.89044

$$
\begin{aligned}
\text{2d term} &= +\quad 0\;.77 \\
\phi' - \phi &= \;\;0°\,24'\,45''.94 \\
\phi &= 45\;\;39\,13\;.89 \\
\phi' &= 46°\,03'\,59''.83
\end{aligned}
$$

2. To find P' by (48):

$$
\begin{array}{rr}
\log\sin Z & 9.5476117 \\
\log u'' & 3.1996888 \\
\log\cos\phi'(\text{ar. comp.}) & \underline{0.1587526} \\
P' - P = 805''.48 & 2.9060531 \\
= \ 0^\circ\, 13'\, 25''.48 & \\
P = 84\ \ 42\ \ 22\ \ .19 & \\
\hline
P' = 84^\circ\, 55'\, 47''.67 &
\end{array}
$$

3. To find Z by (49):

$$
\begin{array}{rr}
\tfrac{1}{2}(\phi+\phi') = 45^\circ 51'\ 36''.86. & \\
\log\sin\tfrac{1}{2}(\phi+\phi') & 9.8559089 \\
\log(P'-P) & \underline{2.9060531} \\
578''.05 & 2.7619620 \\
= 9'\, 38''.05 & \\
339^\circ\, 20\ \ 13\ \ .62 = 180^\circ + Z & \\
\hline
339^\circ\, 10'\, 35''.57 = Z' &
\end{array}
$$

Equations (47), (48), and (50), though only approximately true, are yet sufficiently exact when the distance between the points M and M' is a small arc, for instance, less than twenty miles. For greater distances the solution should be carried to a higher degree of approximation, and formulæ deduced embracing terms which for shorter distances may be neglected. Such a solution, also taken from PUISSANT, will be found in the *Report of the Superintendent of the United States Coast Survey* for 1884, page 323.

APPENDIX.

THE METHOD OF LEAST SQUARES.

167. The Method of Least Squares is a method of eliminating as far as possible the errors of observation. It has for its main object to determine the most probable result from a large number of observations or measurements each of which is subject to error. The processes by which this is effected are deduced by the application of the well-known principles of the theory of Probability, and hence a knowledge of the elementary principles of that theory is pre-supposed of the student.

168. Errors of Observation.—The errors here referred to are the small accidental errors which necessarily affect all human observations, owing to the imperfection of the senses and of the instruments employed, as well as to any unfavorable conditions under which they are used. Such are as likely to be errors in excess as errors in defect, and hence in the long run tend to counteract each other. In this respect they differ from constant errors, which affect all the measurements in the same way, and therefore have no tendency to destroy each other.

Constant errors arise from some peculiarity of the observer or of the instrument, or some abnormal condition under which the observations are made, which leads to an error always in the same direction, and hence their effect is cumulative. For instance, in the measurement of a line, if the standard of length employed should be slightly in error, the influence of this error on the result will not be reduced by any number of repetitions with the same standard. The

effect of the error can only be destroyed or diminished by using different standards. Though they may all be slightly in error, they will be as likely to be too long as too short, and the errors will therefore tend to destroy each other. In this way, by changing observers and instruments, and varying, as far as may be, the methods of observation, the constant errors are changed to accidental ones, and then their effect will be rendered a minimum by the application of the method of least squares. The observations to be discussed should be freed from all constant errors, otherwise the results arrived at will not be the most probable.

Probability of Errors of Observation.

169. Observations are of two kinds: First, *direct*, as when the quantity to be determined is measured directly; and secondly, *indirect*, as when the magnitude directly measured is a function of one or more quantities whose values are required.

Let us denote by z a quantity whose value is to be determined either directly or indirectly by observation. Suppose there to be a series of m measurements, giving the results n, n', n'', etc., and affected with the errors x, x', x'', etc., then we have

$$x = z - n, \quad x' = z - n', \quad x'' = z - n'', \text{ etc.} \qquad (1)$$

Now, experience teaches us something in regard to the law of distribution of errors. We know, for instance, that in observations made with care by practised observers a small error is more likely to occur than a large one, and that very large ones are not liable to be committed, hence the probability diminishes as the error increases; also that errors in excess and errors in defect are equally probable. Thus the probability of an error is a function of the error itself. If we denote by $f(x)$ the probability that the error does not exceed x, the probability that it is comprised between x and $x + dx$ will be

$$f(x+dx)-f(x)=\frac{df(x)}{dx}dx,$$

which becomes for the particular errors x_1, x_2, etc.,

$$\frac{df(x_1)}{dx}dx, \quad \frac{df(x_2)}{dx}dx, \quad \text{etc.}$$

Thus $\frac{df(x)}{dx}$ is a function which expresses the law of probability of errors, and it may be taken to represent the probability of the error x. Denoting it, for brevity, by $\phi(x)$, then $\phi(x)dx$ is the probability that the error is comprised between x and $x+dx$.

170. Form of the Probability Function. — The probability that the error of an observation is between any two limits, as a and b, is the sum of all the elements of the form $\phi(x)dx$ between those limits, that is,

$$\int_a^b \phi(x)\,dx.$$

But it is necessarily between $-\infty$ and $+\infty$, hence we have

$$\int_{-\infty}^{+\infty} \phi(x)\,dx = 1 \qquad (2)$$

since unity is the measure of certainty.

Denote by P the probability that in the observations from which a certain set of values of the unknown quantities were found, the errors x, x', x'', etc., were committed, then since the probability of the occurrence of any number of independent events is the product of their separate probabilities, we have

$$P=\phi(x)\cdot\phi(x')\cdot\phi(x'')\cdot\text{etc.},$$

and

$$\log P=\log\phi(x)+\log\phi(x')+\log\phi(x'')+\text{etc.} \qquad (3)$$

Any other set of values of the unknown quantities will give a different system of errors, and the most probable val-

ues are those whose errors are such as will render P, and therefore $\log P$, a maximum. But in order to find when $\phi(x)$ is a maximum we must know the form of the function. This can best be found by considering a special case, and, as the function is entirely general, the result thus obtained will be applicable to all cases.

In the case of a single quantity observed directly, we may assume as self-evident that if a series of measurements be made under similar circumstances, giving results which, so far as we know, are equally good, the *arithmetical mean* of the observed results is the most probable value of the quantity. We shall therefore have

$$z = \frac{n + n' + n'' + \text{etc.}}{m} \tag{4}$$

whence $\quad n + n' + n'' + \text{etc.} = mz = z + z + z + \text{etc.}$

and $\quad 0 = (z - n) + (z - n') + (z - n'') + \text{etc.},$

or, by (1), $\quad 0 = x + x' + x'' + \text{etc.} \quad (5)$

Now putting the first derivative of $\log P$, equation (3), equal to 0, as the condition for a maximum, we have

$$0 = \frac{d \log \phi(x)}{dx} \cdot \frac{dx}{dz} + \frac{d \log \phi(x')}{dx'} \cdot \frac{dx'}{dz} + \text{etc.}$$

But since, from (1), $dx = dz$, $dx' = dz$, etc., the last equation reduces to

$$0 = \frac{d \log \phi(x)}{dx} + \frac{d \log \phi(x')}{dx'} + \frac{d \log \phi(x'')}{dx''} + \text{etc.}$$

If we put $\quad \dfrac{d \log \phi(x)}{dx} = \psi(x),$

we have $\quad 0 = \psi(x) + \psi(x') + \psi(x'') + \text{etc.},$

which may be written in the form,

$$0 = x\frac{\psi(x)}{x} + x'\frac{\psi(x')}{x'} + x''\frac{\psi(x'')}{x''} + \text{etc.} \tag{6}$$

The comparison of (5) and (6) shows that

$$\frac{\psi(x)}{x}=\frac{\psi(x')}{x'}=\frac{\psi(x'')}{x''}=k, \text{ a constant};$$

that is, $$\frac{d\log\phi(x)}{x\,dx}=k,$$

whence $$d\log\phi(x)=kx\,dx,$$

and integrating, $$\log\phi(x)=\tfrac{1}{2}kx^2+\log c,$$

or $$\phi(x)=ce^{\frac{1}{2}kx^2}.$$

Since the probability diminishes as the error increases, k must be negative. Putting $\frac{1}{2}k=-h^2$, we get

$$\phi(x)=ce^{-h^2x^2} \qquad (7)$$

as the form of the function expressing the law of probability of errors.

171. Determination of the Constant c. — The substitution of the value just found for $\phi(x)$ in (2) gives

$$c\int_{-\infty}^{+\infty}e^{-h^2x^2}\,dx=1 \qquad (8)$$

or putting $hx=t$, whence $dx=\dfrac{dt}{h}$,

$$\frac{c}{h}\int_{-\infty}^{+\infty}e^{-t^2}\,dt=1 \qquad (9)$$

To find the value of this definite integral, put

$$q=\int_0^\infty e^{-t^2}\,dt,$$

then we also have $$q=\int_0^\infty e^{-v^2}\,dv,$$

and $$q^2=\int_0^\infty\int_0^\infty e^{-(t^2+v^2)}\,dt\,dv.$$

Let $v = tu$, whence $dv = t\,du$, since t is regarded as constant in integrating with respect to v. Then we have

$$q^2 = \int_0^\infty \int_0^\infty e^{-t^2(1+u^2)}\, t\, du\, dt$$

$$= \int_0^\infty du \int_0^\infty e^{-t^2(1+u^2)}\, t\, dt.$$

But $\displaystyle\int_0^\infty e^{-t^2(1+u^2)}\, t\, dt = \left[-\frac{e^{-t^2(1+u^2)}}{2(1+u^2)}\right]_0^\infty = \frac{1}{2(1+u^2)}$;

hence $$q^2 = \frac{1}{2}\int_0^\infty \frac{du}{1+u^2} = \frac{\pi}{4},$$

and $$q = \int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}.$$

In the same way we shall find

$$\int_{-\infty}^0 e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2};$$

hence $$\int_{-\infty}^{+\infty} e^{-t^2}\, dt = \sqrt{\pi}.$$

Substituting this in equation (9), we find

$$\frac{c}{h}\sqrt{\pi} = 1, \text{ or } c = \frac{h}{\sqrt{\pi}};$$

and the probability function, equation (7), becomes

$$\phi(x) = \frac{h}{\sqrt{\pi}} e^{-h^2x^2} \tag{10}$$

172. The Measure of Precision. — The constant h may be regarded as a measure of the quality of the observations. It is the same for any two series in which the probability of committing a given error is the same, but different if the probabilities are different, being greatest for the series in which the probability of a given error is least.

In one series, the probability that the error of an observation lies between $-\delta$ and $+\delta$ is

$$\int_{-\delta}^{+\delta} \phi(x)\,dx,$$

or, by (10),

$$\frac{h}{\sqrt{\pi}}\int_{-\delta}^{+\delta} e^{-h^2x^2}\,dx = \frac{1}{\sqrt{\pi}}\int_{-h\delta}^{+h\delta} e^{-h^2x^2}\,d(hx);$$

and in another series the probability that an error lies between $-\delta'$ and $+\delta'$ is

$$\frac{h'}{\sqrt{\pi}}\int_{-\delta'}^{+\delta'} e^{-h'^2x^2}\,dx = \frac{1}{\sqrt{\pi}}\int_{-h'\delta'}^{+h'\delta'} e^{-h'^2x^2}\,d(h'x).$$

These two expressions are equal when $h\delta = h'\delta'$. For example, if $h' = 2h$, the expressions become equal when $\delta' = \frac{1}{2}\delta$; that is, any error has the same probability in the first system as half that error in the second, or the accuracy of the second system is twice as great as that of the first. The quantity h is therefore called the *measure of precision* of the observations.

173. Minimum Squares. — If a series of m observations be made in order to determine the values of one or more unknown quantities, those values are most probable for which the errors committed in the observations have the greatest probability. But the probability of the system of errors x, x', x'', etc., being the product of their separate probabilities, is, by (10),

$$P = \frac{h^m}{\sqrt{\pi^m}} e^{-h^2(x^2+x'^2+x''^2+\text{etc.})} \tag{11}$$

where h is the precision of the series. For the most probable system of errors, P is a maximum, which requires that $x^2 + x'^2 + x''^2 + \text{etc.}$ shall be a minimum; hence the most probable values of the unknown quantities are those for which the sum of the squares of the errors is the least

possible. The method of determining values of the unknown quantities which shall satisfy this condition is for this reason called the "method of least squares."

If the observations from which the unknown quantities are to be determined are of unequal precision, the expression for the probability of the system of errors will be of the form

$$P = \frac{h \cdot h' \cdot h'' \cdot \cdots}{\sqrt{\pi^m}} e^{-(h^2x^2 + h'^2x'^2 + h''^2x''^2 + \text{etc.})},$$

and when P is a maximum, $h^2x^2 + h'^2x'^2 + \text{etc.}$ is a minimum; that is, if each error is multiplied by its precision, the sum of the squares of the products will be the least possible.

The Probability Curve.

174. If we represent the probability function, $\phi(x)$, by y, we have, equation (10),

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2} \tag{12}$$

from which, assuming different values for x, the corresponding values of y can be found, and a series of points thus

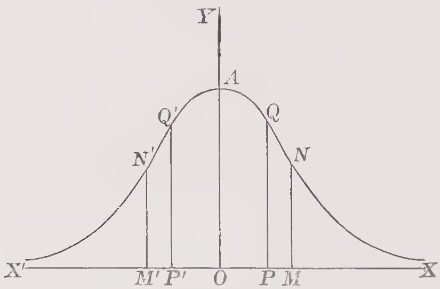

Fig. 19.

determined, the co-ordinates of which will satisfy equation (12). The line joining these points is called the *probability curve.* Its form is represented in Fig. 19.

175. Discussion of the Equation of the Probability Curve. — Since equation (12) involves y to the first power, and x to the second, positive and negative values of x numerically equal give equal values for y, hence the curve is symmetrical with respect to the axis of y.

If $x = 0$, we have

$$y = \frac{h}{\sqrt{\pi}},$$

which is the maximum ordinate, and which varies directly as h. As x increases numerically, y decreases, and when $x = \pm \infty$, $y = 0$.

The first derivative is

$$\frac{dy}{dx} = -\frac{2h^3x}{\sqrt{\pi}} e^{-h^2x^2},$$

from which we see that when $x = 0$, $\frac{dy}{dx} = 0$, that is, the tangent at the vertex is parallel to the axis of x. If $x = \pm \infty$, $y = 0$, and $\frac{dy}{dx} = 0$, hence the axis of x is an asymptote.

The second derivative is

$$\frac{d^2y}{dx^2} = \frac{2h^3}{\sqrt{\pi}} e^{-h^2x^2}(2h^2x^2 - 1),$$

which is zero when $2h^2x^2 - 1 = 0$; hence

$$x = \pm \frac{1}{h\sqrt{2}}$$

are the abscissæ of the points of inflexion.

As the probability of the error x is represented by the ordinate y, the form of the probability curve is seen to agree with the principles already laid down respecting the law of error; showing that the smaller the error the greater its probability, that very large errors have very small

probabilities, and that positive and negative errors of equal magnitude are equally probable.

176. Area of the Probability Curve. — It has already been shown that the probability that an error falls between x and $x + dx$ is

$$\frac{h}{\sqrt{\pi}} e^{-h^2x^2} dx.$$

But we have, equation (8),

$$\int_{-\infty}^{+\infty} y\,dx = \frac{h}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-h^2x^2} dx = 1 \qquad (13)$$

which is the probability that the error is between $-\infty$ and $+\infty$. By the definition of probability, it also represents the whole number of errors, that is, the number of observations in the series. But obviously (13) is the expression for the whole area included between the probability curve and the axis of x; hence this area represents the number of errors in the series, both being taken as unity.

Similarly, $$\frac{h}{\sqrt{\pi}} \int_a^b e^{-h^2x^2} dx$$

is the probability that the error lies between a and b, that is, it is the number of errors to be expected between those limits if the whole number be taken as unity. It is also, evidently, the expression for the area included between the probability curve, the axis of x, and two ordinates drawn at the points $x = a$ and $x = b$. Hence the area of the probability curve between two limits represents the number of errors in the series to be expected between those limits.

Thus the area on the right of OA, the maximum ordinate, represents the number of positive errors, and that on the left of OA the number of negative errors, in the series.

The area $PMNQ$ represents the number of positive errors between OP and OM, and the area $AOPQ$ the number

between 0 and OP. If $OP' = OP$, the area $P'PQQ'$ represents the number of errors numerically less than OP.

177. Computation of the Probability Integral. — If we replace hx by t in the probability integral, we have

$$\frac{h}{\sqrt{\pi}}\int e^{-h^2x^2}\,dx = \frac{1}{\sqrt{\pi}}\int e^{-t^2}\,dt.$$

This integral taken between the limits 0 and t is the number of positive errors less than t. If we consider both positive and negative errors, the number will be twice as great; denoting it by S, we shall have

$$S = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}\,dt \tag{14}$$

which is the number of errors numerically less than t, the whole number of errors being unity. It likewise follows from the last article that S denotes the area of the probability curve between the limits $-t$ and $+t$.

The value of the definite integral in (14) may be found with sufficient approximation if t is small, by developing e^{-t^2} by Maclaurin's Theorem, multiplying by dt and integrating a few of the first terms. We thus find

$$S = \frac{2}{\sqrt{\pi}}\int_0^t dt\left(1 - t^2 + \frac{t^4}{1\cdot 2} - \frac{t^6}{1\cdot 2\cdot 3} + \text{etc.}\right)$$

$$= \frac{2}{\sqrt{\pi}}\left(t - \frac{t^3}{3} + \frac{1}{1\cdot 2}\cdot\frac{t^5}{5} - \frac{1}{1\cdot 2\cdot 3}\cdot\frac{t^7}{7} + \text{etc.}\right),$$

which converges rapidly when t is small.

By successive application of the formula for integration by parts, a series may be obtained for computing this integral for large values of t. The numerical values of S corresponding to successive values of $t = hx$, are given in the following table:

Values of the Probability Integral.

hx	S	hx	S	hx	S
0.0	0.00000	1.0	0.84270	2.0	0.99532
0.1	.11246	1.1	.88021	2.1	.99702
0.2	.22270	1.2	.91031	2.2	.99814
0.3	.32863	1.3	.93401	2.3	.99886
0.4	.42839	1.4	.95229	2.4	.99931
0.5	.52050	1.5	.96611	2.5	.99959
0.6	.60386	1.6	.97635	2.6	.99976
0.7	.67780	1.7	.98379	2.7	.99987
0.8	.74210	1.8	.98909	3.0	0.99998
0.9	0.79691	1.9	0.99279	∞	1.00000

We take from the table for the following values of $t = hx$, the corresponding values of S, and their differences, viz.,

$t = 0.0$	$S = 0.000$	
		.520
$t = \pm 0.5$	$S = 0.520$	
		.323
$t = \pm 1.0$	$S = 0.843$	
		.123
$t = \pm 1.5$	$S = 0.966$	
		.029
$t = \pm 2.0$	$S = 0.995$	
		.004
$t = \pm 2.4$	$S = 0.999$	
		.001
$t = \infty$	$S = 1.000$	

Since the total number of errors is taken as unity, the number of errors in any particular case is found by multiplying the tabular number by the actual number of observations. Thus, in a series of 1000 observations, we may expect

520	errors	between	0	and	± 0.5
323	"	"	0.5	"	1.0
123	"	"	1.0	"	1.5
29	"	"	1.5	"	2.0
4	"	"	2.0	"	2.4
1	"	"	2.4	"	∞

We see that the number of errors comprised between equal intervals decreases very rapidly as t increases, that is, as the errors become greater.

Precision of Observations.

178. The Probable Error. — In theoretical discussions the constant h is used, as we have seen, as a measure of precision of the observations, but for practical purposes it is found more convenient to employ the probable error.

Suppose all the errors of a large series of observations to be arranged in a line in the order of magnitude without regard to their signs, then the error which occupies the middle place in the series is called the *probable error*, and denoted by r. The number of errors less than r being equal to the number greater than r, the probable error may be defined as such a quantity that there is the same probability of the true error being greater, and of its being less, than this.

The probable error is represented geometrically in Fig. 19 by the abscissa OP, of that point on the probability curve whose ordinate, PQ, bisects the area of that part of the curve on the right of the axis of y; which is equivalent to saying that the number of positive errors less than the probable error is equal to the number greater. Since in a large number of observations there is the same number of negative as of positive errors, and they are distributed according to the same law, it is unnecessary to consider that part of the curve on the left of OY, and the abscissa OP will represent the probable error of all the observations of the series.

179. Relation of r and h. — It follows from the definition that the probability that any error of the series is numerically less than r, that is, included between $-r$ and $+r$, is $\frac{1}{2}$. Hence we have for the probable error, equation (14), $S = 0.5$;

and the corresponding value of hx found by interpolation from the table, is

$$hr = 0.47694,$$

whence $$r = \frac{0.47694}{h} \tag{15}$$

Thus the probable error varies inversely as h, and serves equally well to measure the precision of the observations.

179a. The probability that the error of an observation is less than the probable error, r, is, by (14),

$$S = \frac{2}{\sqrt{\pi}}\int_0^{0.47694} e^{-t^2}\,dt = 0.5,$$

and the probability that it is less than nr is found by taking the same integral between the limits 0 and $0.47694\,n$. Thus, the probability that it is less than $\frac{1}{2}r$ is

$$S = \frac{2}{\sqrt{\pi}}\int_0^{0.23847} e^{-t^2}\,dt = 0.264\,;$$

hence in a series of 1000 observations we should expect to find 264 errors less than one-half the probable error, and 500 less than the probable error. If we continue the calculation, we shall find that there will probably be 823 errors less than $2r$, 957 less than $3r$, 993 less than $4r$, and 999 less than $5r$. In series consisting of large numbers of observations it is found that these results of theory agree very nearly with those of experience.

180. Determination of the Value of h. — If there be made a series of m equally good observations, affected with the errors x, x', x'', etc., we have, by (11), for the probability of this system of errors,

$$P = \frac{h^m}{\pi^{\frac{m}{2}}}\,e^{-h^2(x^2+x'^2+x''^2+\text{etc.})}.$$

For the sake of brevity let us put

$$x^2 + x'^2 + x''^2 + \text{etc.} = [x^2],$$

then
$$P = \frac{h^m}{\pi^{\frac{m}{2}}} e^{-h^2[x^2]} \tag{16}$$

The most probable value of h for this series is that value which renders P a maximum. Putting the first derivative of P with respect to h equal to 0, we have

$$\frac{h^{m-1}}{\pi^{\frac{m}{2}}} e^{-h^2[x^2]}(m - 2h^2[x^2]) = 0\,;$$

whence
$$m - 2h^2[x^2] = 0, \qquad h^2 = \frac{m}{2[x^2]},$$

and
$$h = \sqrt{\frac{m}{2[x^2]}} \tag{17}$$

181. The Mean Error. — Another criterion which may be employed to test the accuracy of observations is the *mean error.*

The mean error is the error whose square is the mean of the squares of all the errors. Denoting it by ϵ, we have

$$\epsilon^2 = \frac{[x^2]}{m} \tag{18}$$

and (17) gives
$$h^2 = \frac{1}{2\,\epsilon^2}, \qquad \text{or} \qquad \epsilon^2 = \frac{1}{2\,h^2},$$

and
$$\epsilon = \frac{1}{h\sqrt{2}} = \frac{0.7071}{h} \tag{19}$$

Since, by (15),
$$r = \frac{0.4769}{h},$$

we have
$$\frac{r}{\epsilon} = \frac{0.4769}{0.7071} = 0.6745,$$

and
$$r = 0.6745\,\epsilon \tag{20}$$

Equation (19) shows that the mean error is represented by the abscissa OM, Fig. 19, of the point of inflexion of the probability curve. See Art. 175.

182. Precision of the Arithmetical Mean. — The expression for the probability of a system of m observations of equal precision is,

$$P = \frac{h^m}{\sqrt{\pi^m}} e^{-h^2[x^2]},$$

and for the most probable system, P is a maximum, and hence $[x^2]$ is a minimum. But $[x^2]$ is rendered a minimum by assuming the arithmetical mean, denoted by z_0, of m equally good observations, to be the most probable value; hence P is the probability of this mean if x, x', x'', etc., are the errors of observation compared with z_0. The probability of any other value, as $z_0 + \delta$, will be

$$P' = \frac{h^m}{\sqrt{\pi^m}} e^{-h^2[(x-\delta)^2]} = \frac{h^m}{\sqrt{\pi^m}} e^{-h^2([x^2]-2[x]\delta+m\delta^2)}.$$

Since by (5), $[x] = 0$, and by (18), $[x^2] = m\epsilon^2$, we have

$$P' = \frac{h^m}{\sqrt{\pi^m}} e^{-mh^2(\epsilon^2+\delta^2)},$$

and $$P = \frac{h^m}{\sqrt{\pi^m}} e^{-mh^2\epsilon^2},$$

whence $$P' = Pe^{-mh^2\delta^2}.$$

If $m = 1$, $P' = Pe^{-h^2\delta^2}$, hence h being the measure of precision of a single observation, that of the mean of m such observations is $h\sqrt{m}$. It follows from this that *the precision of the mean of several observations increases as the square root of their number.*

Let ϵ_0 be the mean error, and r_0 the probable error, of the arithmetical mean, then in (15) and (19) putting $h\sqrt{m}$ for h, we shall find

$$\epsilon_0 = \frac{\epsilon}{\sqrt{m}} \tag{21}$$

and

$$r_0 = \frac{r}{\sqrt{m}} \tag{22}$$

183. Determination of Mean and Probable Errors. — The mean and probable errors of a quantity determined directly by observation can be found from the observations themselves. Let z_0 denote the arithmetical mean of m observations, n, n', n'', etc., of some quantity whose true value is z; and let v, v', v'', etc., denote the *residual errors*, that is, $v = z_0 - n$, $v' = z_0 - n'$, $v'' = z_0 - n''$, etc. The residuals v, v', v'', etc., will differ from the true errors x, x', x'', etc., unless z_0 is the true value of z. Suppose v to differ from x by $\pm\delta$, then we shall have

$$x = v \pm \delta, \text{ and } x^2 = v^2 \pm 2v\delta + \delta^2,$$

whence

$$[x^2] = [v^2] \pm 2[v]\delta + m\delta^2 = [v^2] + m\delta^2 \tag{23}$$

since we have, as in (5), $[v] = 0$. The value of δ being unknown, that of $m\delta^2$ cannot be accurately determined: as the best approximation, we may assume δ equal to the mean error, ϵ_0, equation (21); we shall then have

$$m\delta^2 = m\epsilon_0{}^2 = \epsilon^2,$$

whence (23) gives, by (18),

$$m\epsilon^2 = [x^2] = [v^2] + \epsilon^2, \quad \text{and} \quad \epsilon^2 = \frac{[v^2]}{m-1},$$

or

$$\epsilon = \sqrt{\frac{[v^2]}{m-1}} \tag{24}$$

Also, by (20),

$$r = 0.6745\sqrt{\frac{[v^2]}{m-1}} \tag{25}$$

These are the mean and probable errors of any one of the m observations.

These values being substituted in (21) and (22), give

$$\epsilon_0 = \sqrt{\frac{[v^2]}{m(m-1)}} \tag{26}$$

and

$$r_0 = 0.6745\sqrt{\frac{[v^2]}{m(m-1)}} \tag{27}$$

which are the mean and probable errors of the arithmetical mean of m observations.

184. To facilitate the application of (25) and (27), we may put

$$\frac{0.6745}{\sqrt{m-1}} = k, \text{ and } \frac{0.6745}{\sqrt{m(m-1)}} = k';$$

we shall then have

$$r = k\sqrt{[v^2]} \tag{28}$$

$$r_0 = k'\sqrt{[v^2]} \tag{29}$$

The values of k and k' may be taken from the subjoined table:

m	k	k	m	k	k'	m	k	k'
1	. . .	. . .	11	0.21329	0.06431	21	0.15082	0.03291
2	0.67449	0.47694	12	.20337	.05871	22	.14719	.03138
3	.47694	.27536	13	.19471	.05400	23	.14380	.02998
4	.38942	.19471	14	.18707	.05000	24	.14064	.02871
5	.33724	.15082	15	.18026	.04654	25	.13768	.02754
6	.30164	.12314	16	.17415	.04354	26	.13490	.02646
7	.27536	.10408	17	.16862	.04090	27	.13228	.02546
8	.25493	.09013	18	.16359	.03856	28	.12981	.02453
9	.23847	.07949	19	.15898	.03647	29	.12747	.02367
10	0.22483	0.07110	20	0.15474	0.03460	30	0.12525	0.02287

184a. The preceding formulæ have been deduced on the hypothesis of a large number of observations, and the greater

the number, the more nearly will the residuals obtained approximate to the values of the true errors. In the examples which follow, the formulæ are applied to a limited number of observed values simply to illustrate the method, but it must be understood that in order to obtain results of much accuracy the number of observations would have to be greatly increased.

EXAMPLES.

1. An angle was measured thirteen times, giving results as in the first column below. Find its most probable value.

n	v	v^2
62° 42′ 20″.6	0.4	0.16
18 .5	1.7	2.89
20 .7	0.5	0.25
22 .8	2.6	6.76
19 .8	0.4	0.16
20 .4	0.2	0.04
20 .5	0.3	0.09
19 .4	0.8	0.64
19 .5	0.7	0.49
18 .7	1.5	2.25
19 .0	1.2	1.44
21 .2	1.0	1.00
21 .5	1.3	1.69
13)262 .6		$[v^2]=17.86$
20 .2		

The mean of the thirteen measures is 62° 42′ 20″.2. The residuals (v) in the second column are the amounts by which the several numbers in the first column differ from this mean. The sum of the squares of the residuals is 17.86, the square root of which is 4.226. Since $m=13$, we

get from the table, $k = 0.19471$, $k' = 0.05400$; hence (28) and (29) give

$$r = 0.195 \times 4.226 = 0.82;$$
$$r_0 = 0.054 \times 4.226 = 0.23.$$

Hence the most probable value of the angle as determined from these measurements is $62° 42' 20''.2 \pm 0''.23$.

In a similar way may be found the most probable values of the quantities determined by the following observations.

2. *Clock Correction.*	3. *Latitude of Place.*	4. *Meridian Passage.*
$15^s.69$	$17° 45' 8''.7$	$7^h 37^m 16^s.78$
15 .72	6 .7	16 .94
15 .63	6 .7	17 .11
15 .90	7 .3	16 .94
15 .64	8 .6	16 .91
15 .82	9 .4	16 .74
15 .67	6 .1	16 .75
15 .68	7 .3	17 .02
15 .74	8 .6	17 .14
15 .86	6 .2	16 .99
15 .85	9 .0	16 .87
	7 .1	
Ans. $15^s.75 \pm 0^s.02$	$17° 45' 7''.6 \pm 0''.22$	$7^h 37^m 16^s.93 \pm 0^s.03$

In example 4, the given data are the times of passage of a star over the *mean wire* of a transit instrument. They were derived from the observed times of passage over eleven wires, by means of the equatorial intervals. (Arts. 56, 57, 58.)

Weight of Observations.

185. The relative accuracy of two or more observations may be expressed by means of their *weights*. If observations are made under precisely similar circumstances, we

may suppose them to have the same weight; hence the weight depends on the measure of precision, or on the probable error. Since only the relative weights are required, the unit of weight is entirely arbitrary. Let us suppose our observations to be compared with a fictitious observation of the weight *unity*, and let p denote the weight of an actual observation, then the value of p denotes the number of observations of the weight unity which must be combined in order that their arithmetical mean may have the same accuracy as one of the actual observations.

Let r_1 denote the probable error of the fictitious observation, and r' that of an actual observation of the weight p', then according to equation (22),

$$r' = \frac{r_1}{\sqrt{p'}}, \text{ whence } r_1^2 = p'r'^2.$$

If r'' be the probable error of another observation, of the weight p'', we have in the same way,

$$r_1^2 = p''r''^2,$$

hence $$p'r'^2 = p''r''^2, \text{ or } \frac{p'}{p''} = \frac{r''^2}{r'^2};$$

which shows that *the weights of two observations are inversely as the squares of their probable errors;* hence, by equation (15), they are *directly as the squares of their measures of precision.*

186. The Weighted Mean. — To find the mean of a number, m, of observations when their weights are taken into account, suppose n, n', n'', etc., to be the observed results, and p, p', p'', etc., their respective weights, then, by the definition of weight, the quantity n may be considered the mean of p observations of the weight unity; n' the mean of p' observations of the weight unity, etc. Since pn is the sum of the p observations whose mean is n, and so of the others,

the given series of observations of unequal weights is resolved into another series of the weight unity, whose sum is $pn + p'n' + p''n'' +$ etc., and their number $p + p' + p'' +$ etc.; hence, denoting their mean by z_0, we have

$$z_0 = \frac{pn + p'n' + p''n'' + \text{etc.}}{p + p' + p'' + \text{etc.}} = \frac{[pn]}{[p]} \tag{30}$$

which is the expression for the *weighted mean.*

187. Probable Error of the Weighted Mean. — It remains to determine the precision of the weighted mean.

It follows from the definition of weight, that the weight of the mean of m observations of the weight unity will be m; hence the weight of z_0 is $[p]$, and its mean and probable errors are, by (21) and (22),

$$\epsilon_0 = \frac{\epsilon}{\sqrt{[p]}}, \qquad r_0 = \frac{r}{\sqrt{[p]}};$$

in which ϵ and r are the mean and probable errors of an observation whose weight is 1. As there are p residuals expressed by $v = z_0 - n$, p' residuals $v' = z_0 - n'$, etc., we shall have by (24) and (25),

$$\epsilon = \sqrt{\frac{[pv^2]}{m-1}}; \qquad r = 0.6745\sqrt{\frac{[pv^2]}{m-1}};$$

whence

$$\epsilon_0 = \sqrt{\frac{[pv^2]}{(m-1)\,[p]}} \tag{31}$$

$$r_0 = 0.6745\sqrt{\frac{[pv^2]}{(m-1)\,[p]}} \tag{32}$$

which are the mean and probable errors of the weighted mean.

Equation (32) may be written in the form

$$r_0 = k\sqrt{\frac{[pv^2]}{[p]}} \tag{33}$$

the value of k being found in the table on page 132.

EXAMPLES.

1. An angle was measured 40 times, and the results separated into groups as follows: The mean of the first five is taken as a single measurement with the weight 5, the mean of the next eight is taken with the weight 8, and so on, as in the second column below. Find the most probable angle.

p	n	pn	v	v^2	pv^2
5	78° 37′ 50″.0	250″.0	1.0	1.00	5.00
8	48 .3	386 .4	0.7	0.49	3.92
7	48 .9	342 .3	0.1	0.01	0.07
4	49 .2	196 .8	0.2	0.04	0.16
6	49 .3	295 .8	0.3	0.09	0.54
10	48 .9	489 .0	0.1	0.01	0.10
40		1960 .3			9.79

Considering the seconds only, we find by (30),

$$z_0 = \frac{[pn]}{[p]} = \frac{1960''.3}{40} = 49''.0.$$

The residuals in column (v) are the amounts by which the seconds of column (n) differ from z_0. The sum of the products pv^2 is 9.79, and since $m = 6$, (33) gives

$$r_0 = 0.3016\sqrt{\frac{9.79}{40}} = 0.3016 \times 0.49 = 0.149.$$

The most probable value of the angle measured is therefore 78° 37′ 49″ ± 0″.15.

2. *Angle Measurement.*

p	n
3	33° 56′ 13″.4
4	12 .5
6	8 .3
7	10 .0

Ans. 33° 56′ 10″.5 ± 0″.74.

3. *Latitude of Place.*

p	n
10	38° 49′ 41″.8
7	41 .5
8	41 .3
6	41 .9
7	41 .8

38° 49′ 41″.6 ± 0″.08.

188. Determination of Weights. — When the observations are known to have different degrees of precision, and there is no way of finding their probable errors, the relative weights are to be assigned according to the best judgment of the computer. If the mean or probable errors can be computed, the weights will be determined by the principle of Art. 185, that they are inversely as the squares of the probable errors, or of the mean errors.

Example. — The value of a level-scale division was determined by three sets of observations giving the results below. Find the weights of the three results, also their weighted mean and its precision.

Most probable value.	*Mean error.*	*Weight.*
4″.72	0″.0671	222.0
4 .88	0 .0858	135.8
4 .65	0 .0801	155.8

The weights being inversely as the squares of the mean errors, are computed by the formula

$$p = \frac{1}{\epsilon^2}.$$

We can now find the weighted mean and its probable error by (30) and (33).

p	n	pn	v	v^2	pv^2
222.0	4.72	1047.84	0.02	0.0004	2.0888
135.8	4.88	662.70	0.14	0.0196	2.6617
155.8	4.65	724.47	0.09	0.0081	1.2620
513.6		2435.01			4.0125

Hence we have $z_0 = 4''.741$; $r_0 = 0''.04$.

Propagation of Errors.

189. Algebraic Sum of Several Observed Quantities. — Let z_1 and z_2 be two independent observed quantities whose mean errors are ϵ_1 and ϵ_2, and their probable errors r_1 and r_2; and let $Z = z_1 \pm z_2$. It is required to find E, the mean error, and R, the probable error, of Z.

We assume that z_1 and z_2 have been determined by a large number, m, of observations, the true errors of which are, — for z_1, x_1, x'_1, etc.; and for z_2, x_2, x'_2, etc.; then the errors of Z are $x_1 \pm x_2$, $x'_1 \pm x'_2$, etc. We have by (18),

$$m\epsilon_1^2 = [x_1^2], \quad m\epsilon_2^2 = [x_2^2],$$

and

$$mE^2 = (x_1 \pm x_2)^2 + (x'_1 \pm x'_2)^2 + \text{etc.}$$
$$= [x_1^2] \pm 2[x_1 x_2] + [x_2^2].$$

In a large number of observations, there will be as many positive as negative products $x_1 x_2$, hence $[x_1 x_2] = 0$, and

$$mE^2 = [x_1^2] + [x_2^2] = m(\epsilon_1^2 + \epsilon_2^2),$$

whence

$$E = \sqrt{\epsilon_1^2 + \epsilon_2^2} \tag{34}$$

and by (20),

$$R = \sqrt{r_1^2 + r_2^2} \tag{35}$$

Suppose next we have

$$Z = z_1 \pm z_2 \pm z_3.$$

We may consider $z_1 \pm z_2$ as a single quantity, and (34) and (35) will give

$$E = \sqrt{\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2}, \quad R = \sqrt{r_1^2 + r_2^2 + r_3^2};$$

and so on for any number of observed quantities. In general, if we have

$$Z = z_1 \pm z_2 \pm z_3 \pm \text{etc.},$$

we have for the mean and probable errors of Z,

$$E = \sqrt{\epsilon_1^2 + \epsilon_2^2 + \epsilon_2^2 + \text{etc.}} \tag{36}$$

$$R = \sqrt{r_1^2 + r_2^2 + r_3^2 + \text{etc.}} \tag{37}$$

Since, Art. 185, the weights are inversely as the squares of the probable errors, we may put

$$P = \frac{1}{R^2}, \quad p_1 = \frac{1}{r_1^2}, \quad p_2 = \frac{1}{r_2^2}, \text{ etc.};$$

and we shall have, by (37), for the weight of the algebraic sum,

$$P = \frac{1}{r_1^2 + r_2^2 + r_3^2 + \text{etc.}} \tag{38}$$

or

$$P = \frac{1}{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \text{etc.}} \tag{39}$$

In the case of only two observed values, (39) reduces to

$$P = \frac{p_1 p_2}{p_1 + p_2} \tag{40}$$

190. Sum of Multiples of Observed Quantities. — Suppose $Z = az$, in which a is a given constant. Since every observation of z with the error x gives Z with the error ax, the mean error of Z will be $E = a\epsilon$, and its probable error $R = ar$. In general, if we have

$$Z = a_1 z_1 \pm a_2 z_2 \pm a_3 z_3 \pm \text{etc.},$$

then by (36) and (37),

$$E = \sqrt{a_1^2\epsilon_1^2 + a_2^2\epsilon_2^2 + \text{etc.}} \tag{41}$$

$$R = \sqrt{a_1^2r_1^2 + a_2^2r_2^2 + \text{etc.}} \tag{42}$$

Examples.

1. Two angles, AOC and BOC, are measured, giving the results,

$$AOC = 70° \ 39' \ 24''.6 \pm 0''.48,$$
$$BOC = 23 \ \ 08 \ \ 05 \ .3 \pm 0 \ .26.$$

Find their difference, viz., the angle AOB.

We find by (37),

$$R = \sqrt{(0.48)^2 + (0.26)^2} = 0.546,$$

hence we have

$$AOB = 47° \ 31' \ 19''.3 \pm 0''.55.$$

2. A line is measured in three parts, with the following results in feet:

$$42.8 \pm 0.15, \qquad 66.2 \pm 0.22, \qquad 27.3 \pm 0.27.$$

Find the probable length of the whole line.

Ans. 126.3 ± 0.38 ft.

Indirect Observations.

191. Recapitulation. — From what precedes we learn that when a quantity is determined directly by a series of observations all of which are presumed to be of equal weight, its most probable value is found by taking the mean of the observed results; but if the observations are not of equal weight, the most probable value is found, as indicated by (30), by multiplying the result of each observation by its weight, and dividing the sum of the products by the sum of the weights.

192. Equations of Condition. — In the case next to be considered, the unknown quantity is not itself observed directly, but the quantity found by direct observation is a known function of one or more quantities whose values are required. Each observation then establishes an equation between the observed and the unknown quantities, — called an *equation of condition,* — and a number of observations equal to that of the unknowns will suffice to determine their values. But the greater the number of observations, the better results will be obtained, on account of the unavoidable errors with which they are affected; hence the number of equations will generally greatly exceed that of the unknown quantities, and they cannot be solved in the ordinary way. Moreover, as the observations are imperfect, the resulting equations are not strictly true, and no set of values of the unknowns can be found which will satisfy all the equations. Hence the problem is, to combine the equations of condition in such a way as to give the *most probable* values of the unknown quantities, that is, such values as will best represent all the observations.

193. Combining Equations of Condition. — Let us suppose the quantity directly observed to be a linear function of the unknowns x, y, z, etc.; the equations of condition are then of the form

$$ax + by + cz + \text{etc.} + l = 0.$$

But we have just seen that the most probable values of x, y, z, etc., when substituted in the equations of condition will not reduce them exactly to zero, but will leave small residual errors which we may call v, v', v'', etc.; — our equations of condition will then become

$$\left.\begin{array}{l} ax \;\, + by \;\, + cz \;\, + \text{etc.} + l \;\, = v \\ a'x + b'y + c'z + \text{etc.} + l' = v' \\ a''x + b''y + c''z + \text{etc.} + l'' = v'' \\ \qquad\quad \text{etc.} \qquad\qquad \text{etc.} \end{array}\right\} \qquad (43)$$

Now according to Art. 173, the most probable values of x, y, z, etc., are those which substituted in (43) will render $v^2 + v'^2 + v''^2 +$ etc. a minimum. To find the condition for the most probable value of x, denote the sum of all the terms independent of x in the first members by q, q', q'', etc., then equations (43) become

$$\begin{aligned} ax + q &= v \\ a'x + q' &= v' \\ a''x + q'' &= v'' \\ \text{etc.} \quad & \text{etc.} \end{aligned}$$

Squaring both members and adding the results, we have

$$(ax+q)^2 + (a'x+q')^2 + (a''x+q'')^2 + \text{etc.}$$
$$= v^2 + v'^2 + v''^2 + \text{etc.}$$

This is the function to be made a minimum, hence its first derivative must be put equal to zero, which gives

$$a(ax+q) + a'(a'x+q') + a''(a''x+q'') + \text{etc.} = 0 \qquad (44)$$

In like manner we may find similar conditions for the most probable values of the other unknown quantities. From these we deduce the following rule for combining equations of condition: *Multiply each equation of condition by the coefficient of x in that equation, and place the sum of the products equal to zero;* and the same for each of the other unknown quantities.

194. Normal Equations. — The results obtained by applying this rule to the equations of condition are called *normal equations.* Their number is just equal to that of the unknown quantities, and hence they can be solved by the ordinary methods of elimination. The resulting values of x, y, z, etc., satisfy the condition that the sum of the squares of the residual errors is a minimum, and are therefore the most probable values.

195. Observations of Unequal Weight. — If the observations from which the equations of condition are derived are not of equal weight, each equation must be multiplied by the square root of the weight of the observation which furnished it, before applying the rule. The equations are thus reduced to the same unit of weight.

EXAMPLES.

1. Given the following equations of condition, to find the most probable values of x, y, and z.

$$\begin{aligned} x + 2z - 2.9 &= 0, \\ y - z + 2.2 &= 0, \\ x - 3z + 1.1 &= 0, \\ y + 2z - 0.3 &= 0. \end{aligned}$$

From these we find the three normal equations by applying the rule in Art. 193. They are

$$\begin{aligned} 2x - z - 1.8 &= 0, \\ 2y + z + 1.9 &= 0, \\ -x + y + 18z - 11.9 &= 0. \end{aligned}$$

Solving these by elimination, we find

$$x = 1.30, \quad y = -1.35, \quad z = 0.81.$$

2. Given the following equations of condition, to find the most probable values of x, y, and z.

$$\begin{aligned} x - y + 2z - 3 &= 0, \\ 3x + 2y - 5z - 5 &= 0, \\ 4x + y + 4z - 21 &= 0, \\ -x + 3y + 3z - 14 &= 0. \end{aligned}$$

The normal equations are

$$27x + 6y - 88 = 0,$$
$$6x + 15y + z - 70 = 0,$$
$$y + 54z - 107 = 0,$$

the solution of which gives

$$x = 2.47, \quad y = 3.551, \quad z = 1.916.$$

3. The sidereal times of meridian passage of six stars were observed for the purpose of finding the azimuth error of the instrument and the clock correction, by the method of Art. 74. The observed times are given in the first column below.

T	α	$T - \alpha$	δ	$\frac{A}{15}$
$19^h\,00^m\,12^s.96$	$18^h\,59^m\,56^s.42$	$16^s.54$	$13°\,41'\,24''$	0.033
12 46 .97	19 12 31 .05	15 .92	67 27 28	−0.073
19 46 .62	19 30 .06	16 .56	2 52 53	0.043
30 46 .14	30 29 .63	16 .51	−7 17 20	0.051
40 52 .85	40 36 .45	16 .40	10 19 39	0.036
48 50 .09	48 34 .14	15 .95	69 58 13	−0.089

The errors of level and collimation were practically destroyed, hence in equation (19) of Chap. III., we may make $b = 0$, $c = 0$; which reduces it to

$$\alpha = T + \Delta T + aA,$$

in which a, the azimuth constant, is expressed in *time*. If we replace the last term by $\frac{aA}{15}$, a will be expressed in *arc*, and the equation may be written

$$(T - \alpha) + \Delta T + a\frac{A}{15} = 0.$$

Substituting in this the values of $T - \alpha$ and $\frac{A}{15}$ from the table, we obtain the following six equations of condition:

$$16.54 + \Delta T + 0.033\,a = 0,$$
$$15.92 + \Delta T - 0.073\,a = 0,$$
$$16.56 + \Delta T + 0.043\,a = 0,$$
$$16.51 + \Delta T + 0.051\,a = 0,$$
$$16.40 + \Delta T + 0.036\,a = 0,$$
$$15.95 + \Delta T - 0.089\,a = 0.$$

In order to reduce the absolute terms, put

$$e = \Delta T + 16.55\,;$$

the equations will then become

$$-0.01 + e + 0.033\,a = 0,$$
$$-0.63 + e - 0.073\,a = 0,$$
$$+0.01 + e + 0.043\,a = 0,$$
$$-0.04 + e + 0.051\,a = 0,$$
$$-0.15 + e + 0.036\,a = 0,$$
$$-0.60 + e - 0.089\,a = 0.$$

From these we find the two normal equations:

$$-1.42 + 6e + 0.001\,a = 0,$$
$$0.092 + 0.001\,e + 0.02\,a = 0.$$

Multiply the first by 20, and subtract the second, then

$$-28.49 + 120\,e = 0\,; \quad \text{whence } e = 0^{s}.24.$$
$$\Delta T = e - 16.55 = -16^{s}.31 = \text{clock correction.}$$

Substitute the value of e in the second normal equation, then

$$0.092 + 0.0002 + 0.02\,a = 0\,;$$

whence $\quad a = -4''.61 = \text{azimuth error, E. of N.}$

Conditioned Observations.

196. In some cases the results obtained by observation are not independent, but are connected by rigorous conditions which they are required to satisfy. Such observations are said to be *conditioned*. As examples of conditioned observations we may cite the following:

(1) If two or more magnitudes of any kind, and also their sum, are measured, the results must satisfy the condition that the sum of the single magnitudes is equal to the measured sum.

(2) If the angles which close the horizon at a given station are measured, the results are subject to the condition that their sum is 360°.

(3) If the three angles of a triangle are measured, the sum of the results should equal 180° if the triangle is plane, or 180° + the spherical excess if it is spherical.

197. **Adjustment of Observations.** — Now in practice the results of observations are never found to satisfy such conditions exactly, so that it becomes necessary to apply to them small corrections; and these should be so determined as to give the *most probable* values, as well as such as will satisfy the required conditions. The most probable values will be found by distributing the total error among the observations in such way that the correction which each receives is inversely proportional to its weight. If the observations are of equal weight, the observed values will receive equal corrections.

The process of finding and applying these corrections is called *adjustment* of the observations. The following examples include only the simplest cases.

Examples.

1. The differences of longitude of Brest and Paris, France, and Greenwich, England, were determined in 1872 as follows. It is required to adjust the observations.

(1) Brest west of Greenwich,	$17^m\ 57^s.165$,	weight	19.
(2) Greenwich west of Paris,	9 21 .107,	"	6.
(3) Brest west of Paris,	27 18 .199,	"	6.
Sum of (1) and (2) =	27 18 .272		
Total error =	$0^s.073$		

As the corrections are to be inversely as the weights, let that of observation (1) be denoted by $\frac{x}{19}$, and those of observations (2) and (3) by $\frac{x}{6}$; we shall then have

$$\left(\tfrac{1}{19} + \tfrac{1}{6} + \tfrac{1}{6}\right)x = \tfrac{66}{171}x = 0.073,$$

whence $x = 0.189$. The corrections then are

$$\frac{x}{19} = 0.009, \quad \frac{x}{6} = 0.032\,;$$

which being applied to the observed values give for the adjusted values,

(1) $17^m\ 57^s.156$,

(2) 9 21 .075,

(3) 27 18 .231,

which satisfy the condition that the sum of the first two is equal to the third. The correction for the sum will evidently have a sign opposite to those of the single observations.

2. It is required to adjust the following angles measured at the station O:

$$\begin{aligned}
AOB &= 76^\circ\, 27'\, 51''.4, \text{ weight } 6.\\
BOC &= 128\ \ 31\ \ 26\ .8, \quad \text{“} \quad 4.\\
COD &= 69\ \ 16\ \ 15\ .3, \quad \text{“} \quad 5.\\
DOA &= 85\ \ 44\ \ 33\ .7, \quad \text{“} \quad 6.\\
\hline
& 360^\circ\, 00'\, 07''.2
\end{aligned}$$

To find the corrections, we put

$$\left(\tfrac{1}{6} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6}\right)x = \frac{141}{180}x = 7.2,$$

whence $x = 9.19$. Hence the required corrections are

$$\frac{x}{4} = 2.30, \ \ \frac{x}{5} = 1.84, \ \ \frac{x}{6} = 1.53;$$

which being subtracted from the observed results according to their weights, give as the most probable values:

$$\begin{aligned}
AOB &= 76^\circ\, 27'\, 49''.87\\
BOC &= 128\ \ 31\ \ 24\ .50\\
COD &= 69\ \ 16\ \ 13\ .46\\
DOA &= 85\ \ 44\ \ 32\ .17\\
\hline
& 360^\circ\, 00'\, 00''.00
\end{aligned}$$

3. Adjust the following measured angles of a spherical triangle, the weights being equal. The spherical excess is $4''.14$.

Measured values.	*Adjusted values.*
41° 58′ 51″.475	41° 58′ 51″.182
53 53 55 .455	53 53 55 .162
84 07 18 .089	84 07 17 .796
180 00 5 .019	180 00 4 .140
4 .140	
Error = 0 .879	

4. Adjust the following measured angles of a plane triangle.

Measured values.		*Adjusted values.*
55° 12′ 29″.7,	weight 10.	55° 12′ 33″.14
76 31 43 .2,	" 12.	76 31 46 .07
48 15 38 .5,	" 15.	48 15 40 .79
179 59 51 .4		180 00 00 .00
Error = 8 .6		

The corrections are found to be 3.44, 2.87, and 2.29.

5. The following angles closing the horizon were measured at Pine Mount station of the U. S. Coast Survey. Find the adjusted values.

Measured values.		*Adjusted values.*
65° 11′ 52″.500,	weight 3.	65° 11′ 53″.4145
66 24 15 .553,	" 3.	66 24 16 .4675
87 02 24 .703,	" 3.	87 02 25 .6175
141 21 21 .757,	" 1.	141 21 24 .5005
359 59 54 .513		360 00 00. 0000
Error = 5 .487		

TABLES.

EXPLANATION OF THE TABLES.

Table I., from Bessel's Refraction Tables, gives the mean refraction (assuming an atmospheric pressure of 29.6 inches, and temperature of 49° F.) for each degree of altitude from 10° to 75°, and then for every five degrees to the zenith, together with the factors by which it is to be multiplied for different readings of the barometer and thermometer. The following is an example of the method of finding the correction for refraction by means of Table I. :—

Given the apparent altitude = 41° 23′ 17″ = 41°.388 ; Bar. reading, 30.2 in. ; Ther. reading, 57°.

We find from the table —

Mean refraction	= 65″.4	log 1.81558
Bar. 30.2 ; factor	= 1.020	log 0.00860
Ther. 57 ; factor	= 0.984	log $\bar{1}$.99299
Cor. for refraction	= 65″.64	log 1.81717

Table II. gives the logarithms of A and B, the coefficients in the equation of equal altitudes, for values of the interval $2t$, from 3 to 9 hours. For explanation, with an example of the use of this table, see Art. 95.

Table III. gives the values of k, the coefficient of the expression for the reduction to the meridian, for values of the hour angle up to 18 minutes ; also the logarithms of n, the factor depending on the rate of the clock. See Articles 123 and 124, and the example following.

TABLE I.

The Correction for Refraction.

App. Alt.	Mean Ref.	Diff.	App. Alt.	Mean Ref.	Diff.	Bar.	Factor.	Ther.	Factor.
°	″		°	′ ″		Inch.		°	
90	0.0	5.1	**44**	59.7	2.1	**31.0**	1.047	**88**	.929
85	5.1	5.1	**43**	1 1.8	2.2			**86**	.932
80	10.2	5.3	**42**	1 4.0	2.3	**30.9**	1.044	**84**	.935
75	15.5	1.1	**41**	1 6.3	2.4	**30.8**	1.041	**82**	.939
74	16.6	1.1	**40**	1 8.7	2.5	**30.7**	1.037	**80**	.942
73	17.7	1.1	**39**	1 11.2	2.6	**30.6**	1.034	**78**	.946
72	18.8	1.1	**38**	1 13.8	2.7	**30.5**	1.031	**76**	.949
71	19.9	1.1	**37**	1 16.5	2.8	**30.4**	1.027	**74**	.953
70	21.0	1.2	**36**	1 19.3	3.0	**30.3**	1.024	**72**	.956
69	22.2	1.1	**35**	1 22.3	3.1	**30.2**	1.020	**70**	.960
68	23.3	1.2	**34**	1 25.4	3.3	**30.1**	1.017	**68**	.964
67	24.5	1.2	**33**	1 28.7	3.4	**30.0**	1.014	**66**	.967
66	25.7	1.2	**32**	1 32.1	3.7			**64**	.971
65	26.9	1.3	**31**	1 35.8	3.9	**29.9**	1.010	**62**	.975
64	28.2	1.2	**30**	1 39.7	4.1	**29.8**	1.007	**60**	.978
63	29.4	1.3	**29**	1 43.8	4.4	**29.7**	1.003	**58**	.982
62	30.7	1.3	**28**	1 48.2	4.6	**29.6**	1.000	**56**	.986
61	32.0	1.3	**27**	1 52.8	5.0	**29.5**	.997	**54**	.990
60	33.3	1.4	**26**	1 57.8	5.4	**29.4**	.993	**52**	.994
59	34.7	1.4	**25**	2 3.2	5.7	**29.3**	.990	**50**	.998
58	36.1	1.4	**24**	2 8.9	6.3	**29.2**	.987	**48**	1.001
57	37.5	1.4	**23**	2 15.2	6.7	**29.1**	.983	**46**	1.005
56	38.9	1.5	**22**	2 21.9	7.4	**29.0**	.980	**44**	1.009
55	40.4	1.5	**21**	2 29.3	8.0			**42**	1.013
54	41.9	1.6	**20**	2 37.3	8.8	**28.9**	.976	**40**	1.017
53	43.5	1.6	**19**	2 46.1	9.7	**28.8**	.973	**38**	1.022
52	45.1	1.6	**18**	2 55.8	10.8	**28.7**	.970	**36**	1.026
51	46.7	1.7	**17**	3 6.6	12.0	**28.6**	.966	**34**	1.030
50	48.4	1.8	**16**	3 18.6	13.5	**28.5**	.963	**32**	1.034
49	50.2	1.7	**15**	3 32.1	15.3	**28.4**	.960	**30**	1.038
48	51.9	1.9	**14**	3 47.4	17.5	**28.3**	.956	**28**	1.042
47	53.8	1.9	**13**	4 4.9	20.1	**28.2**	.953	**26**	1.047
46	55.7	2.0	**12**	4 25.0	23.5	**28.1**	.949	**24**	1.051
45	57.7	2.0	**11**	4 48.5	27.7	**28.0**	.946	**22**	1.055
44	59.7		**10**	5 16.2				**20**	1.060

TABLE II.

Coefficients of the Equation of Equal Altitudes of the Sun.

$2t$	3^h		4^h		5^h	
	Log A	Log B	Log A	Log B	Log A	Log B
m						
0	9.4172	9.3828	9.4260	9.3635	9.4374	9.3369
2	.4174	.3822	.4263	.3627	.4378	.3358
4	.4177	.3817	.4266	.3620	.4383	.3348
6	.4179	.3811	.4270	.3612	.4387	.3337
8	.4182	.3806	.4273	.3604	.4391	.3327
10	.4184	.3800	.4277	.3596	.4396	.3316
12	.4187	.3794	.4280	.3588	.4400	.3305
14	.4190	.3789	.4284	.3580	.4405	.3294
16	.4193	.3783	.4288	.3572	.4409	.3283
18	.4195	.3777	.4291	.3564	.4414	.3272
20	.4198	.3771	.4295	.3555	.4418	.3261
22	.4201	.3765	.4299	.3547	.4423	.3249
24	.4204	.3759	.4302	.3538	.4427	.3238
26	.4207	.3752	.4306	.3530	.4432	.3226
28	.4209	.3746	.4310	.3521	.4437	.3214
30	.4212	.3740	.4314	.3512	.4441	.3203
32	.4215	.3733	.4317	.3503	.4446	.3191
34	.4218	.3727	.4321	.3494	.4451	.3178
36	.4221	.3720	.4325	.3485	.4456	.3166
38	.4224	.3713	.4329	.3476	.4460	.3154
40	.4227	.3707	.4333	.3467	.4465	.3142
42	.4231	.3700	.4337	.3457	.4470	.3129
44	.4234	.3693	.4341	.3448	.4475	.3116
46	.4237	.3686	.4345	.3438	.4480	.3103
48	.4240	.3679	.4349	.3429	.4485	.3091
50	.4243	.3672	.4353	.3419	.4490	.3078
52	.4246	.3665	.4357	.3409	.4494	.3064
54	.4250	.3657	.4361	.3399	.4500	.3051
56	.4253	.3650	.4366	.3389	.4505	.3038
58	9.4256	9.3643	9.4370	9.3379	9.4510	9.3024

TABLE II.

Coefficients of the Equation of Equal Altitudes of the Sun.

2t	6^h		7^h		8^h	
	Log A	Log B	Log A	Log B	Log A	Log B
m						
0	9.4515	9.3010	9.4685	9.2530	9.4884	9.1874
2	.4521	.2996	.4691	.2511	.4892	.1848
4	.4526	.2982	.4697	.2492	.4899	.1822
6	.4531	.2968	.4704	.2473	.4906	.1796
8	.4536	.2954	.4710	.2454	.4913	.1769
10	.4542	.2940	.4716	.2434	.4921	.1742
12	.4547	.2925	.4723	.2415	.4928	.1715
14	.4552	.2911	.4729	.2395	.4935	.1687
16	.4558	.2896	.4735	.2375	.4943	.1659
18	.4563	.2881	.4742	.2355	.4950	.1630
20	.4569	.2866	.4748	.2334	.4958	.1602
22	.4574	.2850	.4755	.2313	.4965	.1573
24	.4580	.2835	.4761	.2292	.4973	.1543
26	.4585	.2819	.4768	.2271	.4980	.1513
28	.4591	.2804	.4774	.2250	.4988	.1483
30	.4597	.2788	.4781	.2228	.4996	.1453
32	.4602	.2772	.4788	.2206	.5003	.1422
34	.4608	.2756	.4794	.2184	.5011	.1390
36	.4614	.2739	.4801	.2162	.5019	.1359
38	.4620	.2723	.4808	.2140	.5027	.1327
40	.4625	.2706	.4815	.2117	.5035	.1294
42	.4631	.2689	.4821	.2094	.5042	.1261
44	.4637	.2672	.4828	.2070	.5050	.1228
46	.4643	.2655	.4835	.2047	.5058	.1194
48	.4649	.2638	.4842	.2023	.5066	.1159
50	.4655	.2620	.4849	.1999	.5074	.1125
52	.4661	.2602	.4856	.1974	.5082	.1089
54	.4667	.2584	.4863	.1950	.5091	.1054
56	.4673	.2566	.4870	.1925	.5099	.1017
58	9.4679	9.2548	9.4877	9.1900	9.5107	9.0981

TABLE III.

For Computing the Reduction to the Meridian.

Values of k.

P	0^m	1^m	2^m	3^m	4^m	5^m	6^m	7^m
s	″	″	″	″	″	″	″	″
0	0.00	1.96	7.85	17.67	31.42	49.09	70.68	96.20
2	0.00	2.10	8.12	18.07	31.94	49.74	71.47	97.12
4	0.01	2.23	8.39	18.47	32.47	50.40	72.26	98.04
6	0.02	2.38	8.66	18.87	33.01	51.07	73.06	98.97
8	0.03	2.52	8.94	19.28	33.54	51.74	73.86	99.90
10	0.05	2.67	9.22	19.69	34.09	52.41	74.66	100.84
12	0.08	2.83	9.50	20.11	34.64	53.09	75.47	101.78
14	0.11	2.99	9.79	20.53	35.19	53.77	76.29	102.72
16	0.14	3.15	10.09	20.95	35.74	54.46	77.10	103.67
18	0.18	3.32	10.39	21.38	36.30	55.15	77.93	104.63
20	0.22	3.49	10.69	21.82	36.87	55.84	78.75	105.58
22	0.26	3.67	11.00	22.25	37.44	56.55	79.58	106.55
24	0.31	3.85	11.31	22.70	38.01	57.25	80.42	107.51
26	0.37	4.03	11.63	23.14	38.59	57.96	81.26	108.48
28	0.43	4.22	11.95	23.60	39.17	58.68	82.10	109.46
30	0.49	4.42	12.27	24.05	39.76	59.40	82.95	110.44
32	0.56	4.62	12.60	24.51	40.35	60.11	83.81	111.43
34	0.63	4.82	12.93	24.98	40.95	60.84	84.66	112.41
36	0.71	5.03	13.27	25.45	41.55	61.57	85.52	113.40
38	0.80	5.24	13.62	25.92	42.15	62.31	86.39	114.40
40	0.87	5.45	13.96	26.40	42.76	63.05	87.26	115.40
42	0.96	5.67	14.31	26.88	43.37	63.79	88.14	116.40
44	1.06	5.90	14.67	27.37	43.99	64.54	89.01	117.41
46	1.15	6.13	15.03	27.86	44.61	65.29	89.89	118.43
48	1.26	6.36	15.39	28.35	45.24	66.05	90.78	119.45
50	1.36	6.60	15.76	28.85	45.87	66.81	91.68	120.47
52	1.48	6.84	16.14	29.36	46.50	67.58	92.57	121.49
54	1.59	7.09	16.51	29.86	47.14	68.35	93.47	122.53
56	1.71	7.34	16.89	30.38	47.79	69.12	94.38	123.57
58	1.83	7.60	17.28	30.90	48.43	69.90	95.29	124.61

TABLE III.

For Computing the Reduction to the Meridian.

Values of k.

P	8^m	9^m	10^m	11^m	12^m	13^m
s	"	"	"	"	"	"
0	125.65	159.02	196.32	237.54	282.68	331.74
2	126.70	160.20	197.63	238.98	284.26	333.44
4	127.75	161.39	198.94	240.42	285.83	335.15
6	128.81	162.58	200.26	241.87	287.41	336.86
8	129.87	163.77	201.59	243.33	289.00	338.58
10	130.94	164.97	202.92	244.79	290.58	340.30
12	132.01	166.17	204.25	246.25	292.18	342.02
14	133.09	167.37	205.59	247.72	293.78	343.75
16	134.17	168.58	206.93	249.19	295.38	344.49
18	135.25	169.80	208.27	250.67	296.99	347.23
20	136.34	171.02	209.62	252.15	298.60	348.97
22	137.43	172.24	210.98	253.63	300.21	350.71
24	138.53	173.47	212.34	255.12	301.83	352.46
26	139.63	174.70	213.70	256.62	303.46	354.22
28	140.74	175.94	215.07	258.12	305.09	355.98
30	141.85	177.18	216.44	259.62	306.72	357.74
32	142.96	178.43	217.81	261.12	308.36	359.51
34	144.08	179.68	219.19	262.64	310.00	361.28
36	145.20	180.93	220.58	264.15	311.65	363.07
38	146.33	182.19	221.97	265.68	313.30	364.85
40	147.46	183.46	223.36	267.20	314.95	366.64
42	148.60	184.72	224.76	268.73	316.61	368.42
44	149.74	185.99	226.16	270.26	318.27	370.21
46	150.88	187.27	227.57	271.79	319.94	372.01
48	152.03	188.55	228.98	273.34	321.62	373.82
50	153.19	189.83	230.39	274.88	323.29	375.62
52	154.35	191.12	231.81	276.43	324.97	377.43
54	155.51	192.41	233.24	277.98	326.66	379.26
56	156.67	193.71	234.67	279.55	328.35	381.08
58	157.84	195.01	236.10	281.12	330.04	382.90

TABLE III.

For Computing the Reduction to the Meridian.

Values of k. Values of Log n.

P	14^m	15^m	16^m	17^m	Rate.	Log n.
s	"	"	"	"	s	
0	384.74	441.63	502.46	567.2	−30	9.9996985
2	386.56	443.60	504.55	569.4	28	7186
4	388.40	445.56	506.65	571.6	26	7387
6	390.24	447.54	508.76	573.9	24	7588
8	392.09	449.51	510.86	576.1	22	7789
10	393.94	451.50	512.98	578.4	20	7990
12	395.79	453.48	515.09	580.6	18	8191
14	397.65	455.47	517.21	582.9	16	8392
16	399.52	457.47	519.34	585.1	14	8593
18	401.38	459.47	521.47	587.4	12	8794
20	403.26	461.47	523.60	589.6	10	8995
22	405.14	463.48	525.74	591.9	8	9196
24	407.02	465.49	527.89	594.2	6	9397
26	408.90	467.51	530.03	596.5	4	9598
28	410.79	469.53	532.18	598.7	−2	9.9999799
30	412.68	471.55	534.33	601.0	0	0.0000000
32	414.59	473.58	536.50	603.3	+2	0201
34	416.49	475.62	538.67	605.6	4	0402
36	418.40	477.65	540.83	607.9	6	0603
38	420.31	479.70	543.00	610.2	8	0804
40	422.23	481.74	545.18	612.5	10	1005
42	424.15	483.79	547.36	614.8	12	1206
44	426.07	485.85	549.55	617.2	14	1407
46	428.01	487.91	551.73	619.5	16	1608
48	429.93	489.97	553.93	621.8	18	1809
50	431.87	492.05	556.13	624.1	20	2010
52	433.82	494.12	558.34	626.5	22	2212
54	435.76	496.19	560.55	628.8	24	2412
56	437.71	498.28	562.76	631.2	26	2613
58	439.67	500.37	564.98	633.5	+28	0.0002814

681

12823